Announcements

- Quiz on Thurs – up to end of last week
  - 2 stages: Individual and group
  - Group work: teams of 4, and require laptop
- WeBWork Assignment 4 has questions that will help you study
- Text also has questions at end of chapters

Using calculus to sketch the graph of a function

UBC Math 102

Rough sketch

- Practice your skills at getting a rough sketch of any function. Consider asymptotic behaviour for large and small x, find zeros, vertical asymptotes, etc
- Only then apply your calculus tools

First derivative

- Increasing and decreasing functions

1. If $f'(x) > 0$ then $f(x)$ is **increasing**.
2. If $f'(x) < 0$ then $f(x)$ is **decreasing**.

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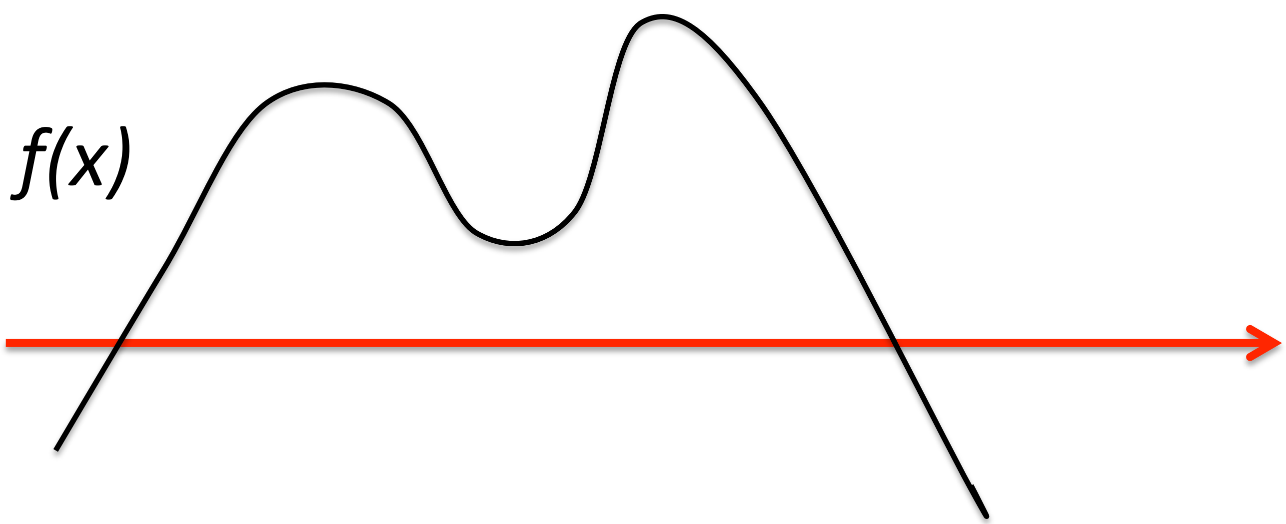

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First derivative

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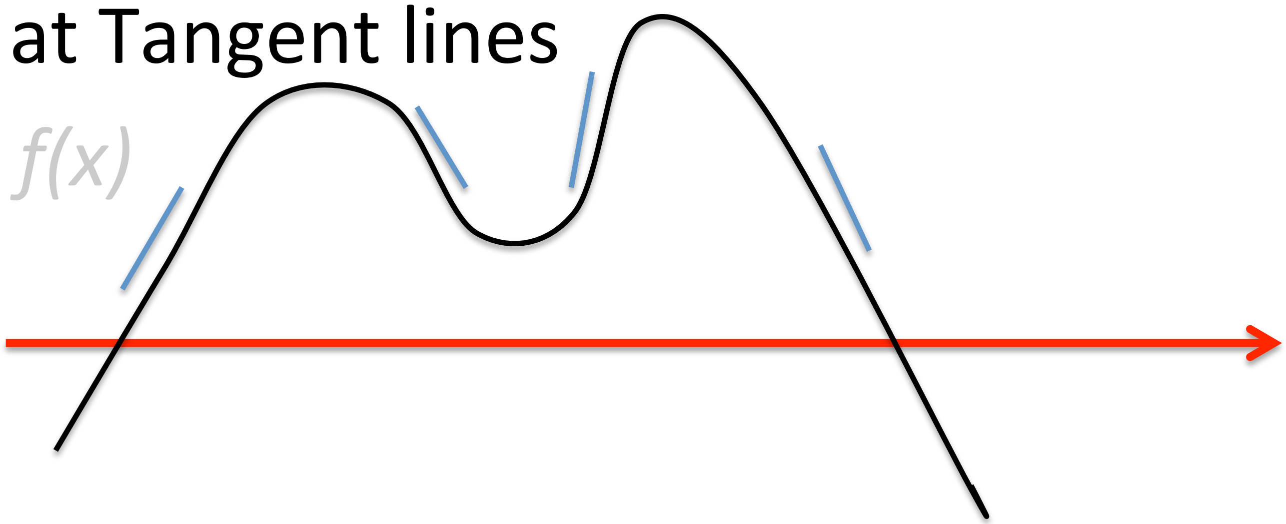
- Look at Tangent lines
-



First derivative

- Increasing and decreasing functions

1. If $f'(x) > 0$ then $f(x)$ is **increasing**.
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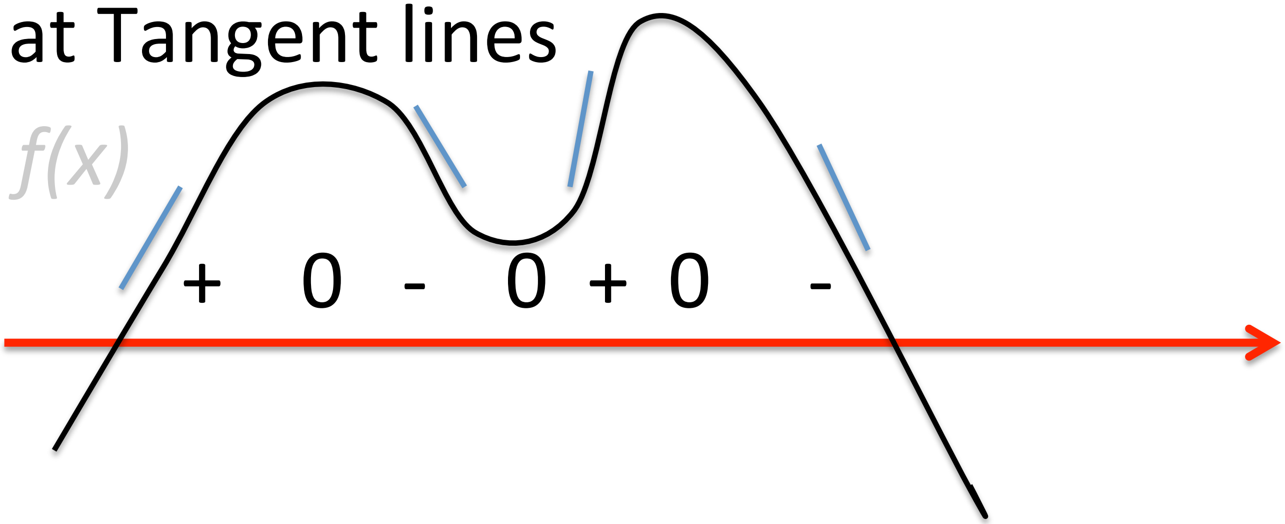
- Look at Tangent lines
-

f ‘(x):



First derivative

- Increasing and decreasing functions

1. If $f'(x) > 0$ then $f(x)$ is **increasing.**
2. If $f'(x) < 0$ then $f(x)$ is **decreasing.**

- Values of x at which f '(x) = 0 are called critical points

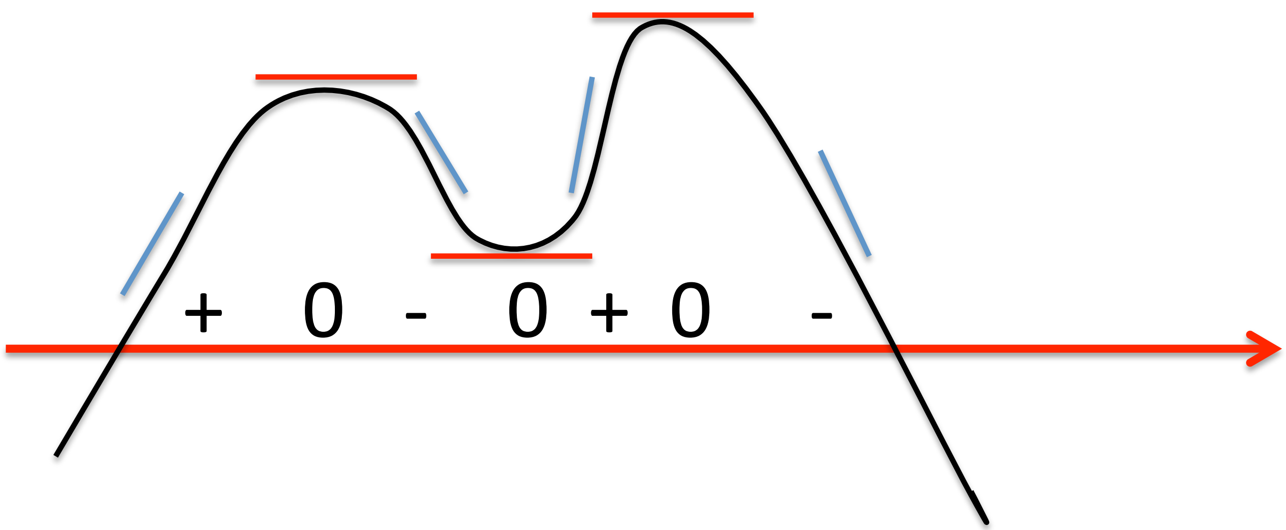
- Values of x at which f '(x) does not exist are also of interest (cusps/discontinuities).

Critical points

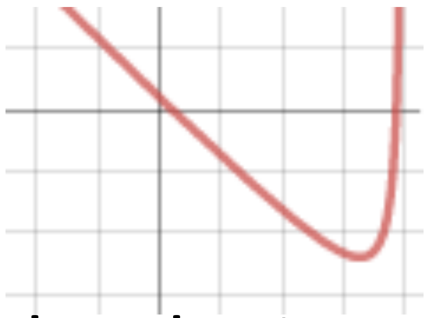
- Values of x at which f ’(x) = 0 are called critical points

- *f(x)*

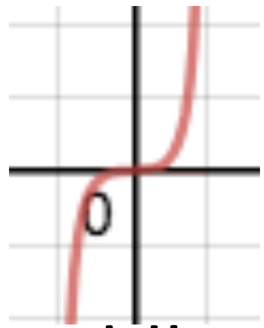
f ‘(x):



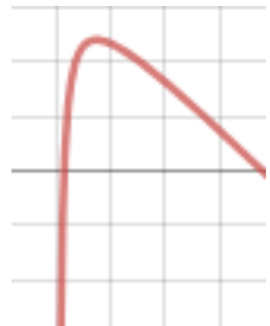
All kinds of critical points



local min



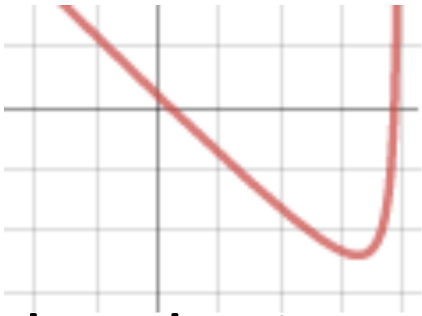
saddle



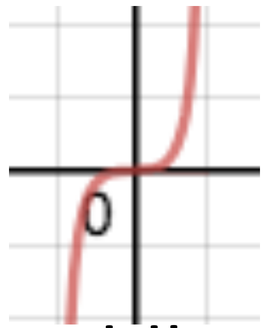
local max

All these graphs have a point at which f’(x)=0

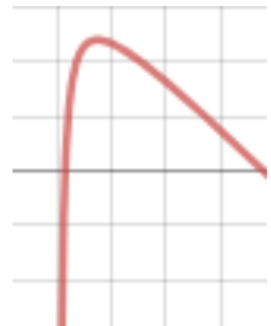
Checking the type of CP



local min



saddle

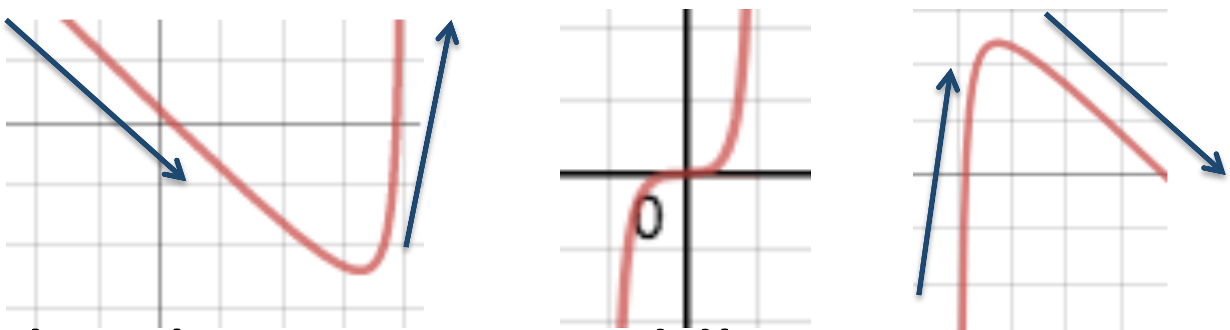


local max

x

Checking f'(x) nearby: are you going uphill or down hill as you move along the x axis?

Sign of 1st derivative



local min | saddle | local max

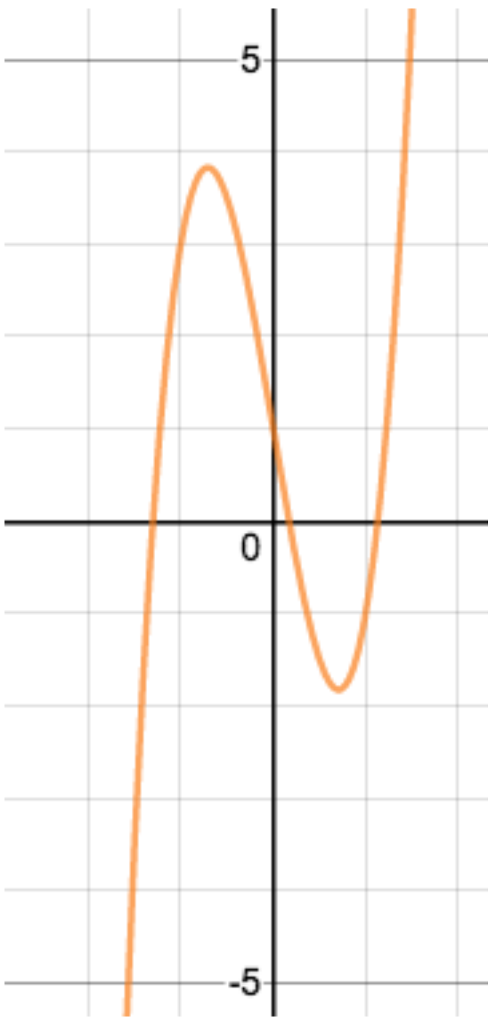
$f'(x)$ $-\ 0\ +$ $+\ 0\ +$ $+\ 0\ -$

x

This is the "First derivative test".

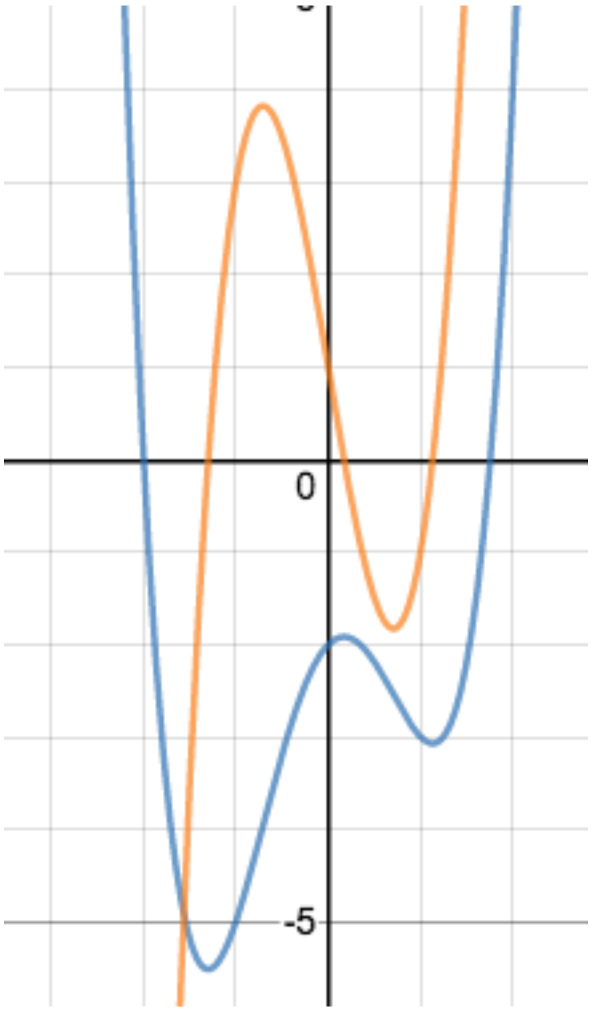
1. If this is f'(x), how many and what types of CPs?

A. two: local max, local min
B. Two: local min, local max
C. Three: local min, max,min
D. Three: local max, min, max
E. Three: local min, saddle, min



Here is the original function

C. Three: local min, max, min



Derivatives are functions too!

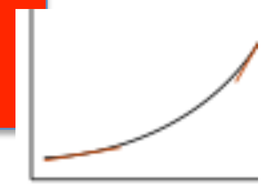
- Given a function f(x)
- Then f'(x) depends on x and is a function too
- Its derivative is f '' (x), the 2nd derivative and this is also a function

The second derivative f''(x)

- (Tells us whether the first derivative is increasing or decreasing)

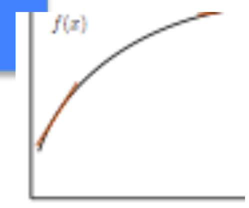
If $f''(x) > 0$ then $f'(x)$ is **increasing**.

Slope is getting more positive



If $f''(x) < 0$ then $f'(x)$ is **decreasing**.

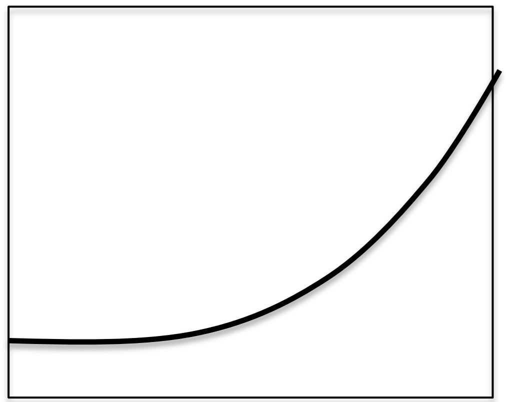
Slope is getting more negative



The second derivative f’’(x)

$f''(x) > 0$

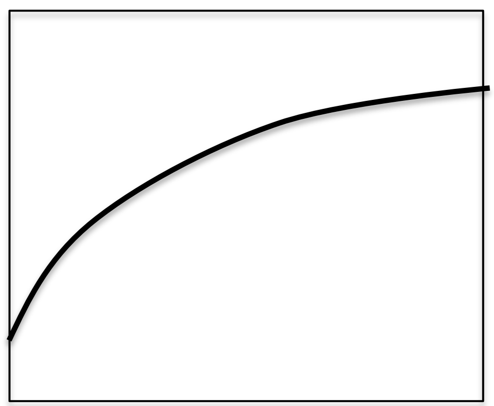
Concave up



Slope is getting more positive

$f''(x) < 0$

Concave down

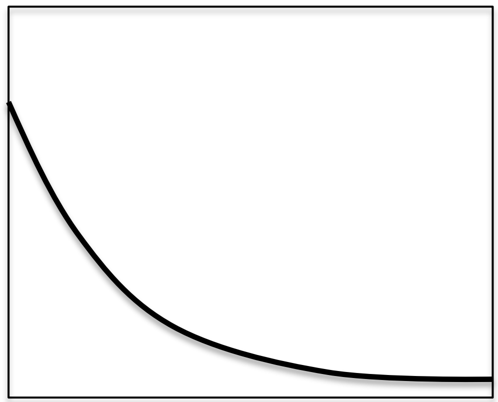


Slope is getting more negative

The second derivative f''(x)

$f''(x) > 0$

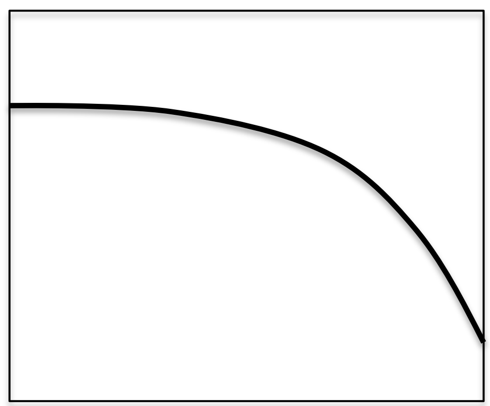
Concave up



Slope is getting more positive

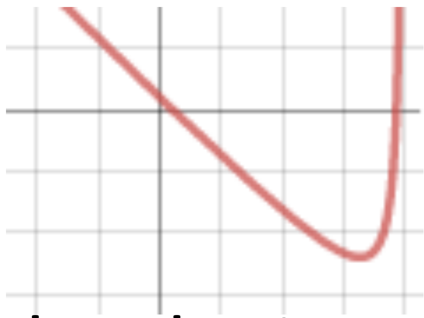
$f''(x) < 0$

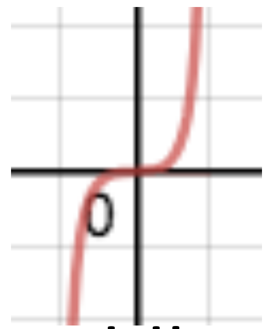
Concave down



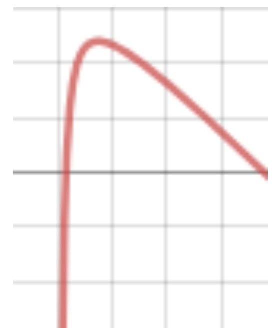
Slope is getting more negative

Checking the type of CP (2)


local min



saddle

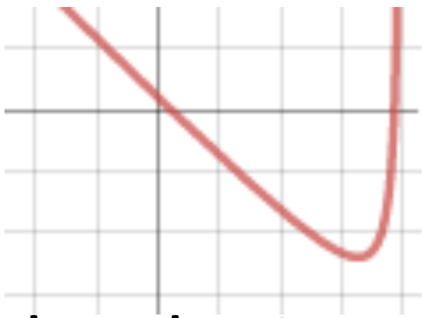

local max

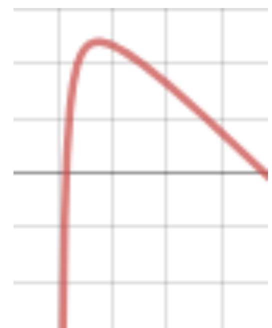
x

Checking f’’(x) nearby:

is graph concave up or down at the CP?

Checking the type of CP (2)
Second derivative test





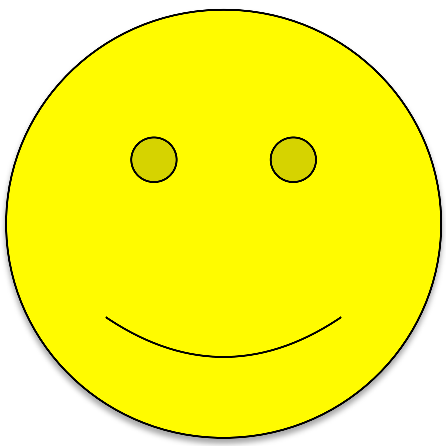
local min	?	local max
$f''(x) > 0$	$f''(x) = 0$	$f''(x) < 0$

Checking $f''(x)$ nearby:

is graph concave up or down at the CP?

Never forget

- I'm always upbeat and positive!
- I'm always downcast and negative..



$f''(x) > 0$

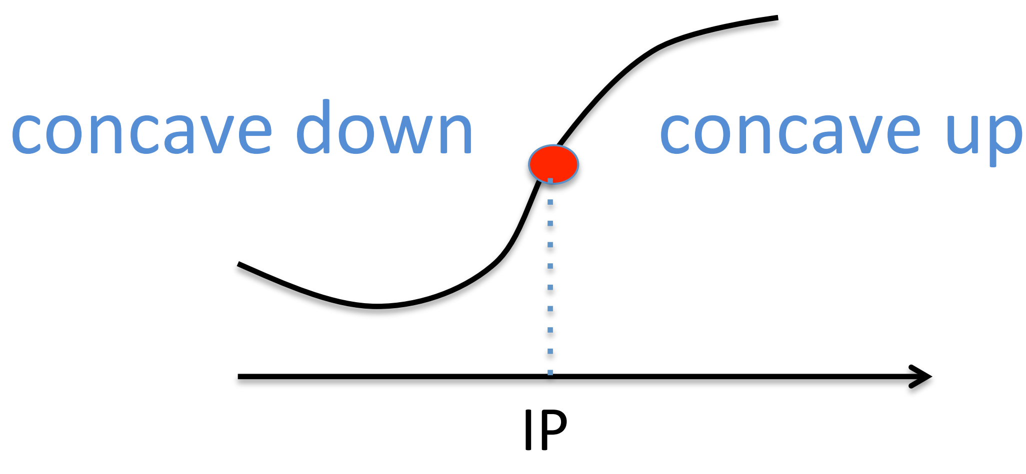
$f''(x) < 0$

Inflection points

Definition:

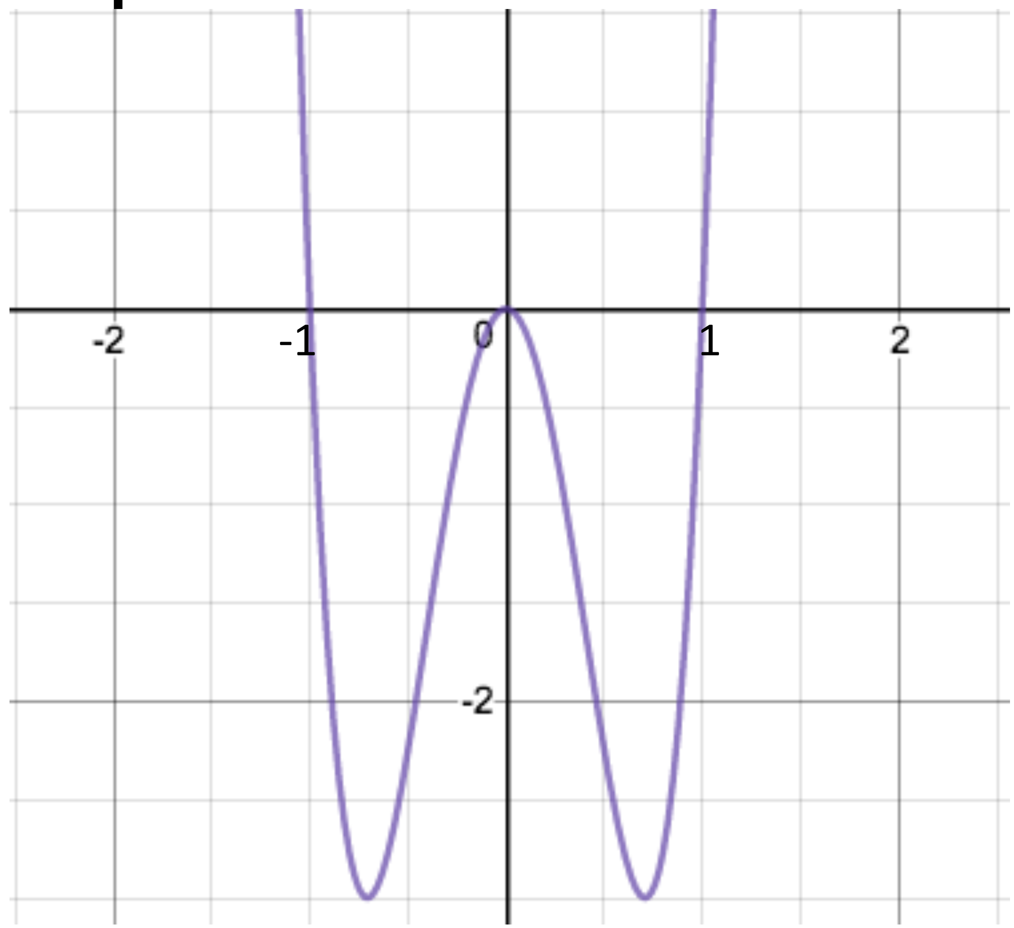
An inflection point is a point at which the concavity of the function changes.

→An inflection point is a place where $f''(x)$ changes sign.



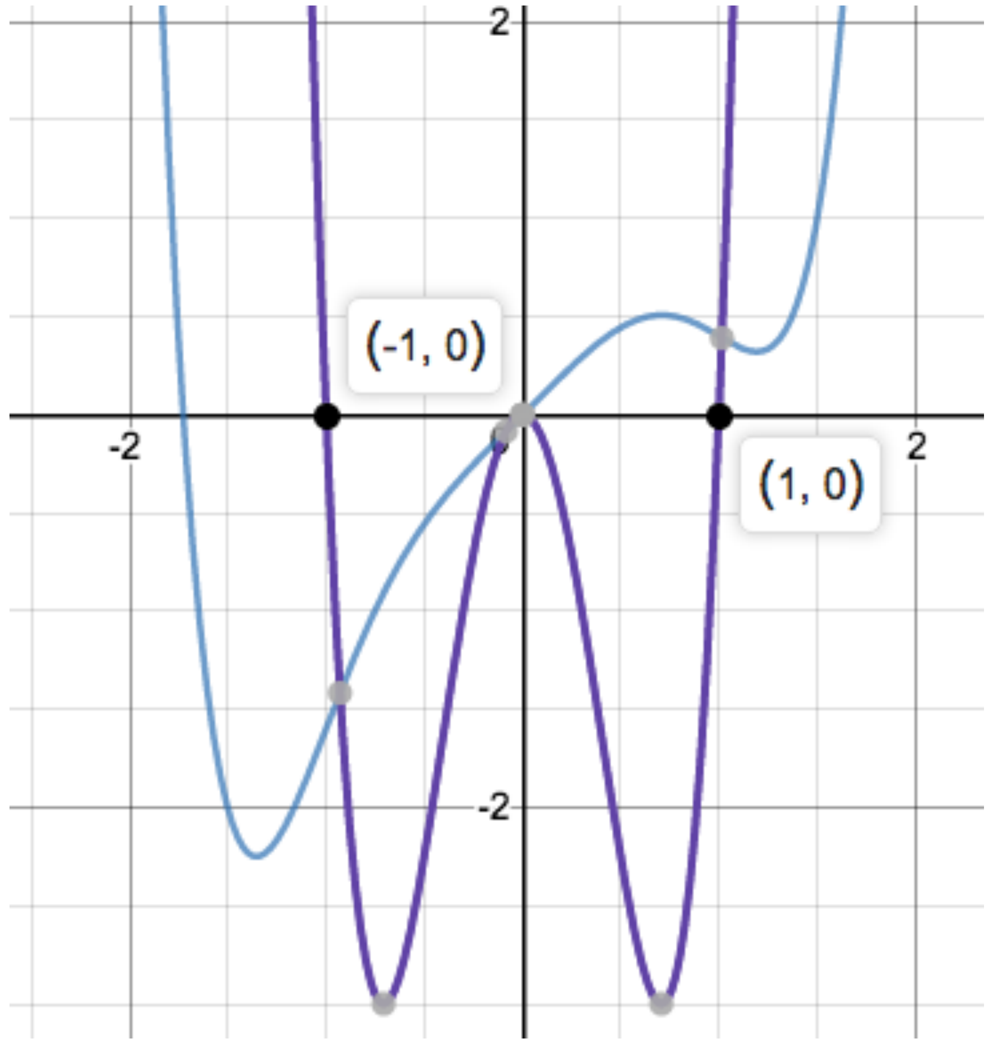
2. If this is f''(x), where are the inflection points?

- A. at -1, 1
- B. at -1, 0, 1
- C. at 0
- D. at -0.74, 0, 0.74
- E. at -0.74, 0.74



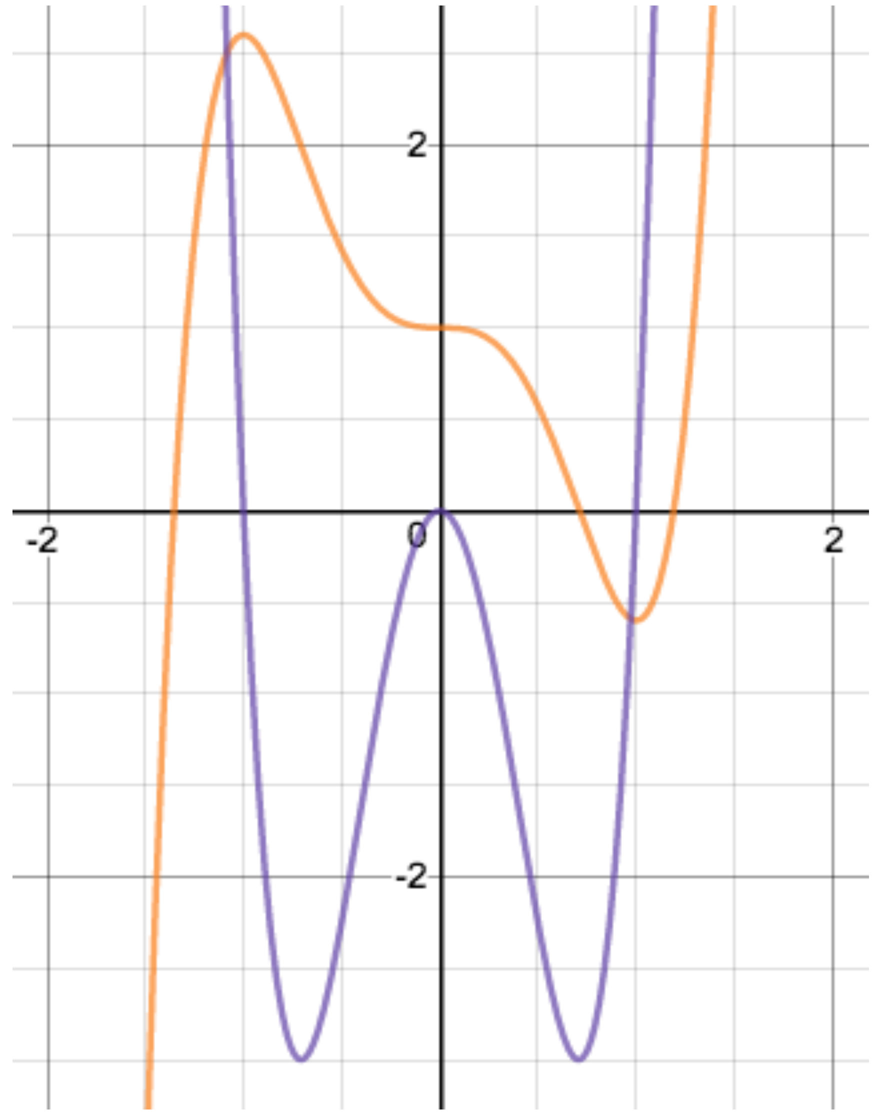
Only two inflection points!

- f''(x) only changes sign at two places!!
- That's where the original function has inflection points



Inflection points are places where f'(x) has a local min or local max!

- Places where f''(x)=0 are critical points of the 1st deriv, f'(x) !



The family: f(x), f’(x), f’’(x)

-

(3) Inflection points

Which of the following is true?

(A) Both $y=x^4$ and $y=x^3$ have inflection points at $x=0$.

(B) Both $y=x^4$ and $y=x^3$ satisfy $f''(x)=0$ at $x=0$.

(C) Both (A) and (B) are true.

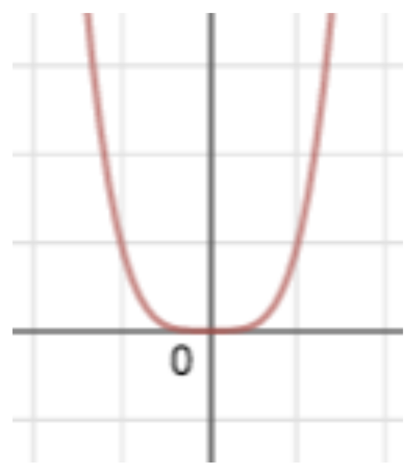
(D) Neither (A) nor (B) is true.

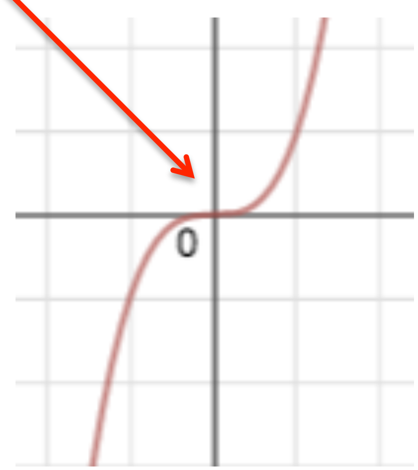
Inflection points

Graphs of $y=x^4$ and $y=x^3$

Both have $f''(0)=0$!

Only one has an inflection point





(4) Inflection point: if f''(x) can be factored into the form

$$f''(x) = x^2(1-x)(x-2)^2(x^2-9)$$

Then the function would have inflection point(s) at

(A) $x = 0, 1, 2, 3$

(B) $x = 1, 2$ only

(C) $x = 1$ only

(D) $x = 1, 3$ only

(E) $x = 1, \pm 3$ only

(5) Find inflection points of the function $f(x) = 3x^5 - 5x^4$

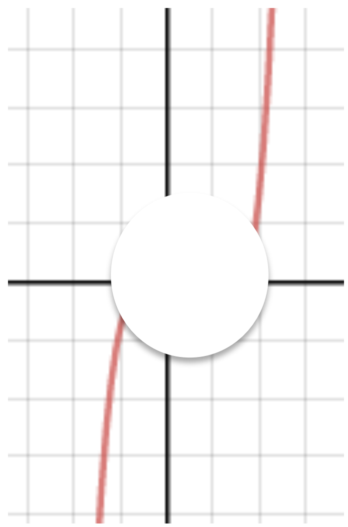
- (A) x=0
- (B) x=1
- (C) x=0, 1
- (D) x=0, -1
- (E) No inflection points

Sketch

$$f(x) = 3x^5 - 5x^4$$

For small x:

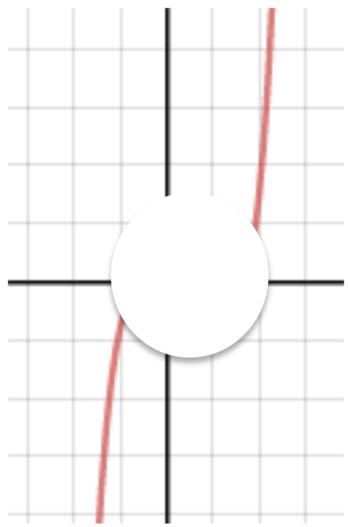
Large x:

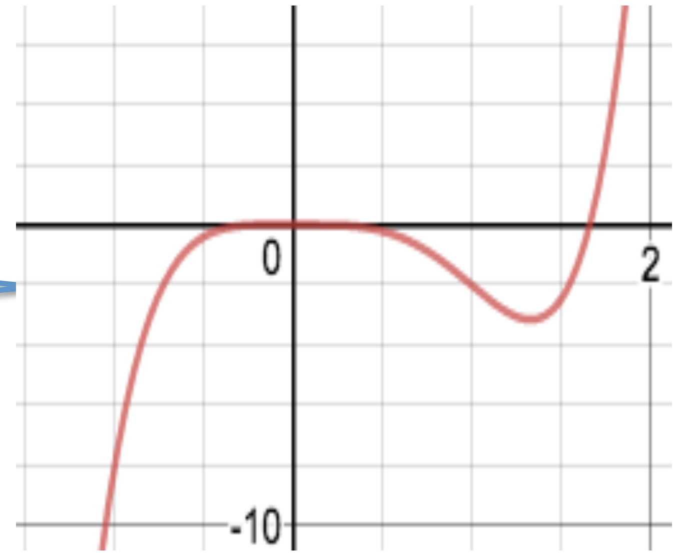


Sketch $f(x) = 3x^5 - 5x^4$

For small x:

Large x:





From the derivatives:

- Function: $f(x) = 3x^5 - 5x^4$
- 1st derivative: f'(x)= $15\,x^4 - 20\,x^3$
- 2nd derivative: f''(x) = $60\,x^2\,(x-1)$

(changes sign only at x=1 !!) IP at x=1 only !!

(6) Suppose $f'(x)<0$, $f''(x)<0$.

Then we can conclude that

(A) The function is constant and concave up.

(B) The function is decreasing and concave down.

(C) The function is decreasing and concave up.

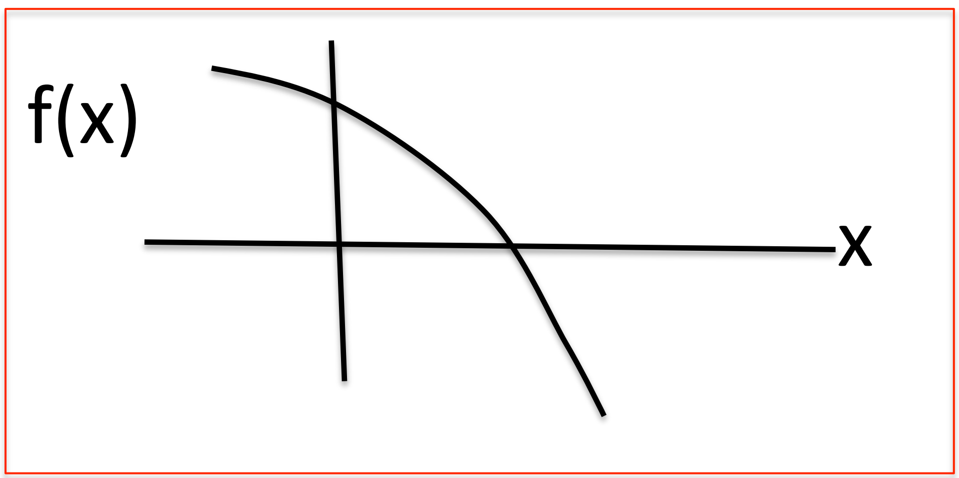
(D) The function is increasing and concave down.

(E) There is a local maximum.

(6) Suppose $f'(x)<0$, $f''(x)<0$.

Then we can conclude that

f(x) is decreasing and concave down



(7) If $f'(x_0)=0$ and $f''(x_0)>0$

Then we can conclude that

(A) The function is increasing and has an inflection point at x_0

(B) The function is decreasing and concave up.

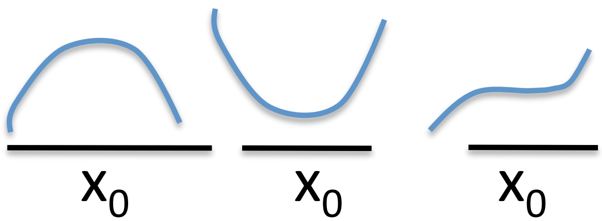
(C) There is a local max at x_0

(D) There is a local min at x_0

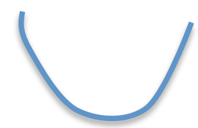
(E) None of the above.

(7) If $f'(x_0)=0$ and $f''(x_0)>0$

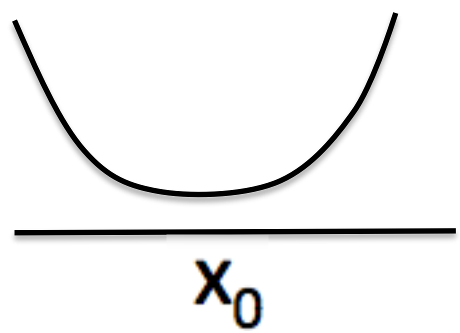
- Critical point



concave up



➔ **Local min at x_0**



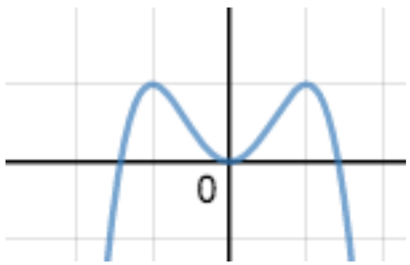
Try out some problems

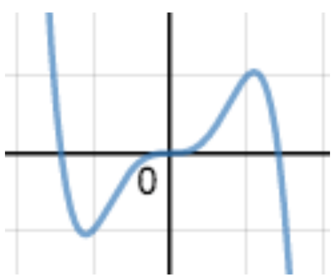
(Worksheets – try working on each one on your own; if stuck, talk to your neighbors..)

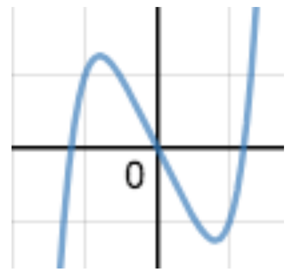
8. What does the function look like?

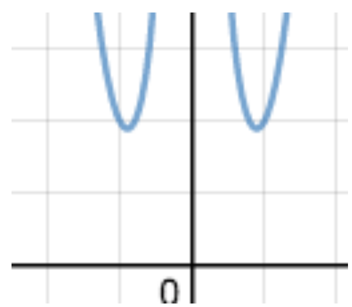
$$f(x) = x^4 - 2x^2$$

- A 

- B 

- C 

D. 

E. 

Where are its critical points? And what types?

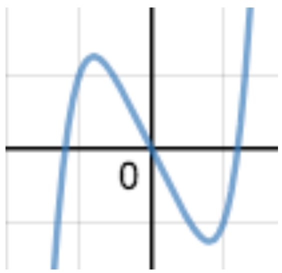
$$f(x) = x^4 - 2x^2$$

- CP: local max at x=0, local min at $x = \pm 1$
- Inflection points at $x = \pm\sqrt{3}/3.$

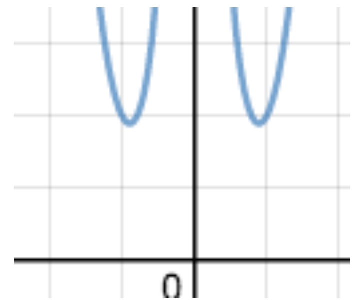
9. What does the function look like?

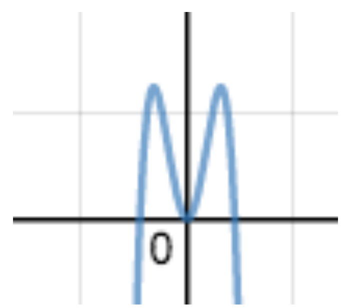
$$f(x) = x^2 + \frac{4}{x}.$$

- A 

- B 

- C 

D. 

E. 

Absolute max

$$f(x) = x^2 + \frac{4}{x}.$$

- On the interval $1 \leq x \leq 3$

There is a local minimum only!

Evaluate at endpoints of interval!

(10) Critical points

Let $$y = f(x) = x^3 + 3x^2 + ax + 1$$

For what range of values of a are there no critical points?

(A) $a < 3$, (B) $a > 3$, (C) $a < 36$,

(D) $a > 36$, (E) $-6 < a < 6$,

Solution

Find $y' = f'(x) = 3x^2 + 6x + a$

Solve $3x^2 + 6x + a = 0$

$$x = \frac{-6 \pm \sqrt{36 - 4a \cdot 3}}{6}$$

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}}$$

$$36 - 12a < 0$$

$$a > 3$$

Answers

- 1. C
- 2. A
- 3. B
- 4. E
- 5. B
- 6. B
- 7. D
- 8. A
- 9. C
- 10.B

Problems to test your skill

Critical points (Dec 2004 exam)

(6) Consider the polynomial

$$y = p(x) = K(x^2 - 16)(x - a)$$

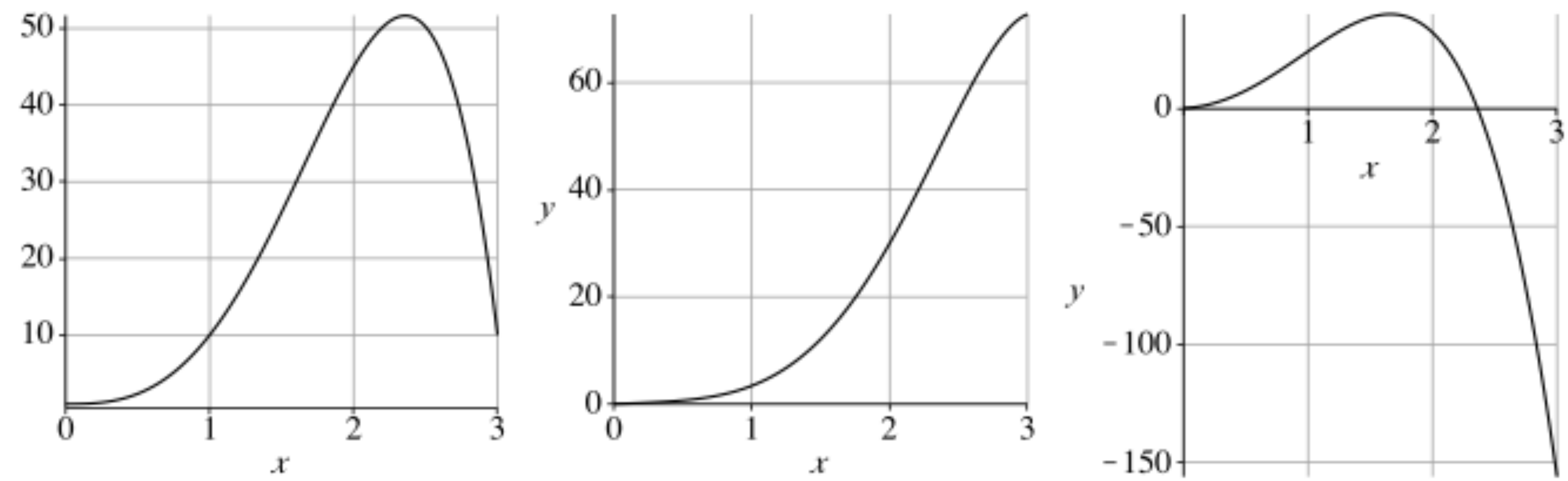
where a, K are constants. Suppose that $p(0) = 12$ and that $p(x)$ has an inflection point at $x = 1/3$.

(a) What are the values of the constants a and K?

(b) Where does this polynomial have critical points?

(c) Which critical point is a maximum?

Which is the distance, velocity, acceleration?



Function and tangent

[10] **8.** A certain function $f(x)$ satisfies the following conditions: $f(1) = 2$, $f'(1) = -1$.

a) What is the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$?

b) Find an approximation for $f(1.01)$.

Now assume in addition that $f''(x) > 0$ for all x.

c) Sketch the graph of $y = f(x)$ in the neighbourhood of $x = 1$.

d) Will your approximation in (b) be larger or smaller than the actual value of $f(1.01)$? Why?

Solutions to previous problems

First try out the problems at the end of the last lecture slides. Only then should you “peek” at the answers.

Solution – part 1:

Find a linear approximation that provides a rough estimate for the value of $(1.1)^8$. Explain why the approximation is an (over/under)-estimate.

- Use $f(x)=x^8$ and base point $x=1$ at which $f(1)=1$.
- Then $f'(x)=8x^7$, and so $f'(1)=8$
- So $f(x)$ is approximated by

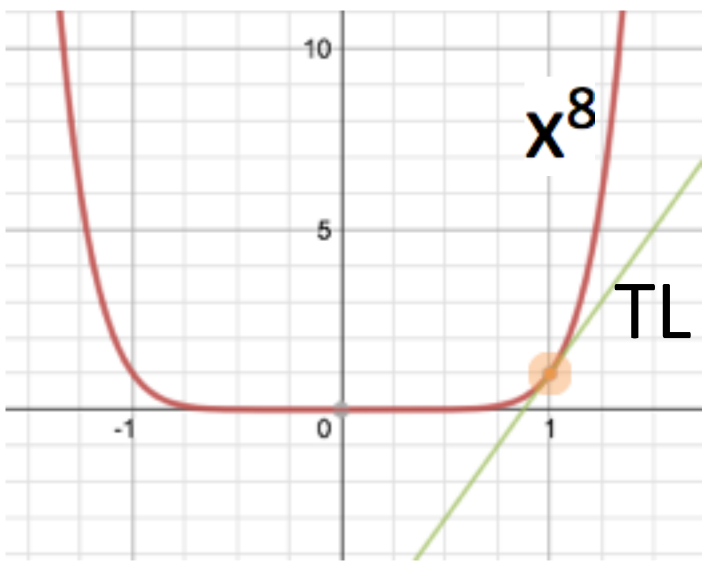
$$f(1)+f'(1)(x-1)=1+8(x-1)$$

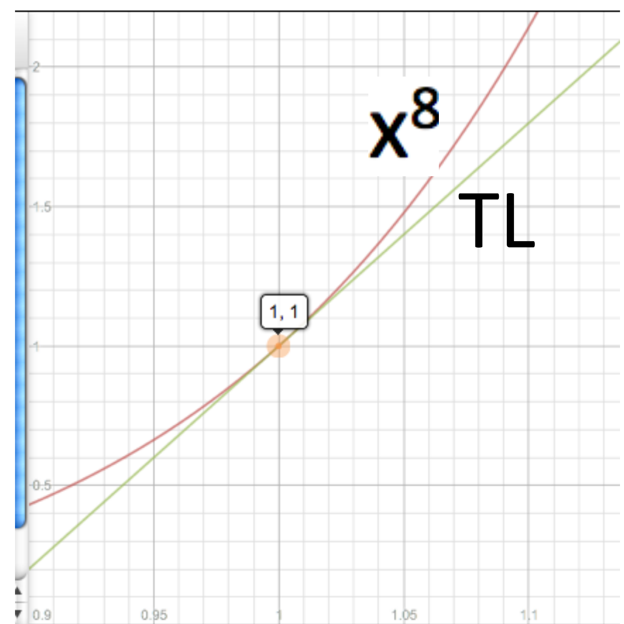
- At $x=1.1$ this leads to

$$(1.1)^8 \approx 1+8(1.1-1)=1.8$$

Solution- Part 2:

$f(x)=x^8 \approx f(1)+f'(1)(x-1)$





TL is below graph of function so linear approx is an underestimate

Find critical point(s) of

$$g(x) = x^5 - 4x^4 + 3x^3 + x^2 - 3x.$$

- **Solution to Problem 5.16:**

  First we have to get the derivative of the function, which we find using the power rule:

$$f(x) = g'(x) = 5x^4 - 16x^3 + 9x^2 + 2x - 3$$

The problem reduces to finding a value of x for which $f(x) = 0$.

We need both $f(x_0), f'(x_0)$, that is we use

$$f'(x) = 20x^3 - 42x^2 + 18x + 2.$$

We set up the spreadsheet using

$$x_0 = 2$$

In the next cell we use the formula

$$x_1 = x_0 - \frac{f(x)}{f'(x)} = x_0 - \frac{5x_0^4 - 16x_0^3 + 9x_0^2 + 2x_0 - 3}{20x_0^3 - 42x_0^2 + 18x_0 + 2}$$

Use Newton's Method to solve this cubic eqn