Tangent lines, cont’d

Linear approximation and Newton’s Method
Last time:

A “challenging” tangent line problem, because we had to figure out the point of tangency.

(A) I get it!
(B) I think I see how we did it
(C) I’m not sure
(D) I’m confused
(E) I’m totally lost
Last time:

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See Section 5.6, Try problem 5.10
Last time

“Use linear approximation to calculate $\sqrt{105}$ ”

Solution:
- figure out the function
- Figure out a point nearby where we know the value of the function
- Use linear approximation
Solution:

- figure out the function \( f(x) = \sqrt{x} \)
- Figure out a point nearby where we know the value of the function. \( \text{We know } \sqrt{100} = 10 \)
- Use linear approximation

\[
 f(x) \approx f(x_0) + f'(x_0)(x - x_0).
\]
Geometry:

“Stick a tangent line” on the function at $x_0=100$, read the value on TL nearby (at $x=105$).
Linear Approx:

- Advantage: easy to calculate:
  \[ f(x) \approx f(x_0) + f'(x_0)(x - x_0). = \sqrt{100} + \frac{1}{2\sqrt{100}}5, \quad = 10 + \frac{5}{20} = 10.25 \]

- One step method

- Disadvantage: not very accurate further away
Another way (Newton’s method)

Convert the problem to the form $f(x)=0$

Use intuition or desmos to get a rough estimate for the zero of the function.
Use Newton’s method to get better and better approximation.

Disadv: Need some intuition
Advantage: Outstanding level of accuracy!
1. Convert to right form

Find decimal approx to \( x = \sqrt{105} \).

To use Newton’s method I would convert this to solving the problem \( f(x) = 0 \) where

A. \( f(x) = x^2 - 105 \)

B. \( f(x) = x - \sqrt{105} \)

C. \( f(x) = x - 10.5 \)

D. \( f(x) = \sqrt{105} \)
Convert to right form

Find decimal approx to $x=\sqrt{105}$.

- Same as: find $x$ such that $x^2=105$
  
  $x^2-105=0$

- Now the problem is in the form $f(x) = 0$

  for $f(x) = x^2-105$
2. My “initial guess” for the zero of 
\[ f(x) = x^2 - 105 \]
would be

A. \( x_0 = 100 \)
B. \( x_0 = 10 \)
C. \( x_0 = 105 \)
D. \( x_0 = \sqrt{105} \)
E. I’m lost.
Initial guess

Find value of $x$ somewhere around zero of $f(x) = x^2 - 105$

That will be the starting value for Newton’s method.

$x_0 = 10$
Newton’s Method

A way to find decimal approximation for the zero of a function

UBC Math 102
Plan

• What is Newton’s Method
• The geometry behind it
• A practical example where we need to use it
• How to use a spreadsheet to implement Newton’s Method.
What is Newton’s Method?

Solve $f(x) = 0$
(3) Which of these is problems is not suitable for Newton’s method?

- (A) Solve $f(x)=0$ for $x$.
- (B) Find the zeros of a function $f(x)$.
- (C) Find where the graph of $f(x)$ crosses the $x$ axis.
- (D) Approximate the value of a function close to a known point $x_0$.
- (E) Find roots of an equation $g(x)=C$. 
Newton’s method can be used to

• (A) Solve $f(x)=0$ for $x$.
• (B) Find the zeros of a function $f(x)$.
• (C) Find where the graph of $f(x)$ crosses the x axis.

(E) It can also find roots of $F(x)=g(x)-C=0$
Hammer and nail

Solve \( f(x) = 0 \)
The geometry behind it
Tangent line

• Equation of tangent line at $x_0$:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

• Find the point $x_1$ at which the tangent line intersects the $x$ axis.
(4) The tangent line intersects the x axis at:

(A) \[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

(B) \[ x_1 = x_0 - \frac{f'(x_0)}{f(x_0)} \]

(C) \[ x_1 = x_0 + \frac{f(x_0)}{f'(x_0)} \]

(D) \[ x_1 = x_0 + \frac{f'(x_0)}{f(x_0)} \]

(E) \[ x_1 = f(x_0) + f'(x_0)(x - x_0) \]
Deriving Newton’s Formula

• Eqn of TL:
  \[ y = f(x_0) + f'(x_0)(x - x_0) \]

• TL intersects x axis when
  \[ y = 0. \]

\[
0 = f(x_0) + f'(x_0)(x - x_0) \quad \Rightarrow \quad (x - x_0) = -\frac{f(x_0)}{f'(x_0)}
\]

\[
\Rightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.
\]

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.
\]
Improving the rough estimate..

• Repeat the process:
  \[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \]
  \[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \]
  \[ \vdots \]

• At the k’th step:
  \[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \]
Put it on a spreadsheet!

Let the spreadsheet do those repetitive calculations for you!

Goal: find decimal approx to square root of 105 accurate to 10 decimal places!
Spreadsheet set up

- The function is $f(x) = x^2 - 105$
- We need its derivative $f'(x) = 2x$
- Initial guess $x_0 = 10$

- Formula to repeat:
  
  $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
Spreadsheet

• The next few slides show how to solve this problem on a spreadsheet.
### Setup

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
</tr>
<tr>
<td>2</td>
<td>( f(x_0) )</td>
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<td></td>
<td>( =A2^2-105 )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
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<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
<td>( f(x_0) )</td>
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<tr>
<td>2</td>
<td>( 10.00000000000000 )</td>
<td>( -5.00000000000000 )</td>
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Transfer $x_1$ to Column A

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<tr>
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<th>B</th>
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<td>10.0000000000000000</td>
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</table>

Highlight the cells and drag down

It should look like:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
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<th>D</th>
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<td>1</td>
<td>$x_0$</td>
<td>$f(x_0)$</td>
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<tr>
<td>2</td>
<td>10.0000000000000000</td>
<td>-5.0000000000000000</td>
<td>20.0000000000000000</td>
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<td>3</td>
<td>10.2500000000000000</td>
<td>0.062500000000000000</td>
<td>20.5000000000000000</td>
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</tbody>
</table>
Drag down the entire row to repeat
Part II:
Looking at all this geometrically
Recap

• Let us look again at linear approximation and at Newton’s method

• We will use desmos and the function

\[ f(x) = \left( 4 + x - \frac{x^3}{27} \right) \]

To illustrate concepts.
PART 1

• A demo of linear approximation
Graph the function: \[ f(x) = \left( 4 + x - \frac{x^3}{27} \right) \]

Use desmos to graph its derivative
It should look like:

\[ f(x) = 4 + x - \frac{x^3}{27} \]

\[ f'(x) \]
Desmos cont’d

Add the value $x_0=5$

A slider should appear for $x_0$ so you can shift the location of that tangent line.

Add the point $(x_0, f(x_0))$ to your graph.
\[ f(x) = 4 + x - \frac{x^3}{27} \]

\[ f'(x) \]

\[ x_0 = 5 \]

\[ (x_0, f(x_0)) \]

(5, 4.37)
Desmos cont’d

Add the equation of a generic tangent line at 
\( x = x_0 \)

You should be able to “animate” \( x_0 \) and see that line sweep across your curve.
It should look like:

\[ f(x) = 4 + x - \frac{x^3}{27} \]

\[ f'(x) \]

\[ x_0 = 5 \]

\[ f(x_0) + f'(x_0)(x - x_0) \]

(5, 4.37)
Using Desmos

Consider the tangent line at \( x_0 = 5 \). Use your graph to find the values below:

\( f(5), f(6) \) and the linear approx to \( f(6) \) based at \( x_0 = 5 \)
(5) Using Desmos

I got the values.

f(5), f(6) and the linear approx to f(6)

(A) 4.37, 4, 5
(B) 4.37, 5, 2
(C) 4.37, 2, 2.5
(D) huh??
PART 2

• A demo of Newton’s Method on Desmos
On desmos

Use **Newton’s Method** to find the largest zero of the function

\[
f(x) = \left(4 + x - \frac{x^3}{27}\right)
\]

Start with rough initial guess \(x_0=5\).
Tangent line at \( x_0 = 5 \)

- TL and \( f(x) \)

\[
f(x) = \left( 4 + x - \frac{x^3}{27} \right)
\]
Newton’s formula

- Add the formula
  \[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

- Plot the point \((x_1,0)\)
Look like this: (here is $x_1$)

Actual zero of the function – we want to find an accurate decimal expansion for it.
Desmos does the work

We got Desmos to compute $x_1$ using the Newton’s Method formula:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

• (We got this by finding where a generic tangent line crosses the x axis, see previous lectures)
Repeat the process

To get closer to the zero of f:

• Construct a tangent line at the point $x_1$
• Find the point where the new tangent line intersects x axis (Find $x_2$).

https://www.desmos.com/calculator/1bcshd1qot
We use the same idea to find $x_2$:

$$f(x_1) + f'(x_1)(x - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$x_2 = 6.70406122137$
To Notice:

• We are getting closer to the actual zero of the function.

UBC Math 102
Effect of initial guess

Go to the slider for x0 and shift it to the value 3.

What happens to the tangent lines?

What happens to the approximation for the zero of the function?
Bad initial guess!

- An initial guess $x_0=2.3$ leads astray! Does not converge to the zero of $f$! See below:
Important comments:

1. The Desmos demonstration is only to show the GEOMETRY that explains how Newton’s Method works. You do not need this (elaborate) construction in a practical problem.

2. How do I find an initial guess? Use Desmos, or sketch the function. Ensure there is no max/min between $x_0$ and the true zero. We will see how to sketch functions in future lectures.
Answers

• 1. A
• 2. B
• 3. D
• 4. A
• 5. C
Solutions to previous problems

First try out the problems at the end of the last lecture slides. Only then should you “peek” at the answers.
Differentiate, using power or quotient rule

Solution to Problem 4.1:

(d) \( f(x) = (x - 1)(x^2 + x + 1) \),

(e) \( f(x) = \frac{x^2 - 9}{x^2 + 9} \),
(d) 
\[ f'(x) = (1)(x^2 + x + 1) + (x - 1)(2x + 1) \]
\[ = x^2 + x + 1 + 2x^2 - 2x + x - 1 \]
\[ = 3x^2 \]

(e) 
\[ f'(x) = \frac{2x(x^2 + 9) - (x^2 - 9)(2x)}{(x^2 + 9)^2} \]
\[ = \frac{2x^3 + 18x - 2x^3 + 18x}{(x^2 + 9)^2} \]
\[ = \frac{36x}{(x^2 + 9)^2} \]
Calculation we did in class

Find the equation of the tangent line to the graph of \( f(x) = x^3 - 3x + 1 \) at \( x=0 \). Where does the tangent line intersect the \( x \)-axis?
Solution: (a) Eqn of tangent line

function: \( y = f(x) = x^3 - 3x + 1 \)

\( f'(x) = 3x^2 - 3 \)

at \( x = 0 \)

\( f(0) = 1 \) \( \rightarrow \) y value of pt on curve. \( (0, 1) \)

\( f'(0) = -3 \) \( \rightarrow \) slope of T.L is \( m = -3 \)

Eqn of T.L:

\[
\frac{y - y_0}{x - x_0} = m \quad \frac{y - 1}{x - 0} = -3
\]

\( y - 1 = -3x \)

\( y = 1 - 3x \)
Related test question (MT1, 2014)  
(Tangent lines)

Consider the function \( f(x) = \frac{3}{x - 2} \).

(a) At which points \((a, f(a))\) does the graph of this function have tangent lines parallel to the line \( y = -x \).

(b) What is the equation of the tangent lines at each of these points.
Solution:

(a) \[ f'(x) = \frac{0 \cdot (x-2) - 3}{(x-2)^2} = \frac{-3}{(x-2)^2} \]
\[ f'(a) = \frac{-3}{(a-2)^2} = -1 \]
\[ a^2 - 4a + 4 = 3 \]
\[ a^2 - 4a + 1 = 0 \]
\[ a = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} \]
\[ f(a) = \frac{3}{a-2} = \frac{3}{\pm \sqrt{3}} = \pm \sqrt{3} \]

1. \((2 + \sqrt{3}, \sqrt{3})\)
2. \((2 - \sqrt{3}, -\sqrt{3})\)
Solution

(b) \[ y = \sqrt{3} - (x - (2 + \sqrt{3})) \]
\[ y = -x + 2 - 2\sqrt{3} \]
Solution to Related test-type problem from last time:

Find a linear approximation that provides a rough estimate for the value of $(1.1)^8$. Explain why the approximation is an (over/under)-estimate.

• (Note: a calculator value is 2.1435)
Test question (Quotient rule, rational functions) MT1 2014

At an all-you-can-eat buffet, the total calories you gain can be represented by the function

\[ E(t) = \frac{At}{b + t} \]

where \( t \geq 0 \) is the time in minutes you spend at the restaurant and \( A \) and \( b \) are positive constants.

• (a) If you stayed for a long time, what asymptote would your total caloric gain approach?
• (b) After how much time do you gain exactly half of that asymptotic caloric amount?
• (c) At time \( t \), what is the instantaneous rate at which your caloric gain changes?
Solution:

(a) $E(t) \rightarrow \sqrt{A}$ as $t \to \infty$

(b) At $t=b$, $E(b) = \frac{Ab}{b+b} = \frac{Ab}{2b} = \frac{A}{2}$

(c) $E'(t) = \frac{A(b+t) - At}{(b+t)^2} = \frac{Ab}{(b+t)^2}$
Test your skill on these problems
Linear Approximation

Find a linear approximation that provides a rough estimate for the value of $(1.1)^8$. Explain why the approximation is an (over/under)-estimate.
Find the prey density such that $P(x) = G(x)$

$$P(x) = \frac{30x^3}{(20^3 + x^3)}$$

$$G(x) = 0.5 \cdot x$$

Note: this problem is similar to the aphid–ladybug problem from week 1 but the function $P(x)$ is not the same.
Newton’s Method

5.16. **Using Newton’s method to find a critical point.** Consider the function

\[ g(x) = x^5 - 4x^4 + 3x^3 + x^2 - 3x. \]

Critical points of a function are defined as values of \( x \) for which \( g'(x) = 0 \). However, for this fifth order polynomial, it is not easy to find such points analytically (i.e., using pencil and paper).

(a) Use Newton’s Method to find a critical point for positive values of \( x \). Find an initial approximation for the critical point by plotting the function, but use a spreadsheet and explain how you set up the calculations. Provide an answer accurate to 8 decimal points.