Inverse Trigonometric functions

And the

"Helen of Mathematics"

UBC Math 102

2000 December Q9 Related Rates

A wheel of radius 1 meter rolls on a flat surface without slipping. The wheel moves from left to right, rotating clockwise at a constant rate of 2 revolutions per second.

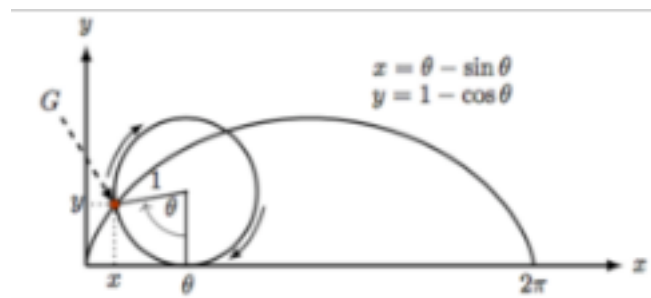
Stuck to the rim of the wheel is a piece of gum, (labeled G); as the wheel rolls along, the gum follows a path shown by the wide arc (called a "cycloid curve*) in the diagram.*

The (x, y) coordinates of the gum (G) are related to the wheel's angle of rotatior θ by the formulae

$$x = \theta - \sin\theta,$$

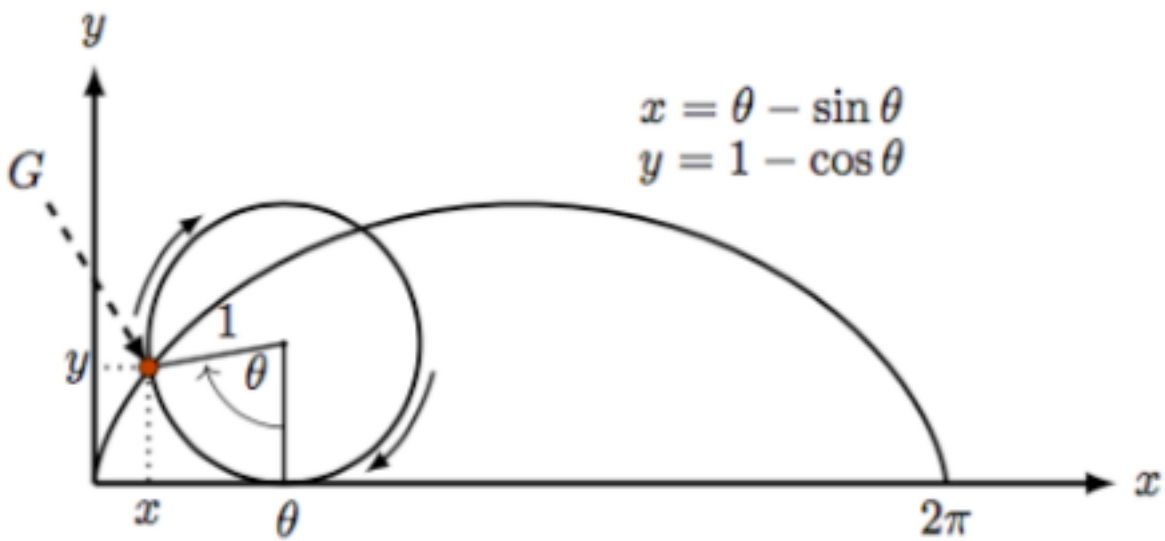
$$y = 1 - \cos\theta,$$

where $0 \leq \theta \leq 2\pi$.



How fast is the gum moving horizontally at the instant that it reaches its highest point? How fast is it moving vertically at that same instant?

The cycloid.

- 

Solution of the Exam problem

- Se Appendix (Document camera work)

Does this curve have an x-y equation?

- Is it possible to “eliminate the angle theta” and write this as a single equation for x and y?

- Yes, and this is one motivation for inverse trig functions.

Inverse Trigonometric functions

UBC Math 102

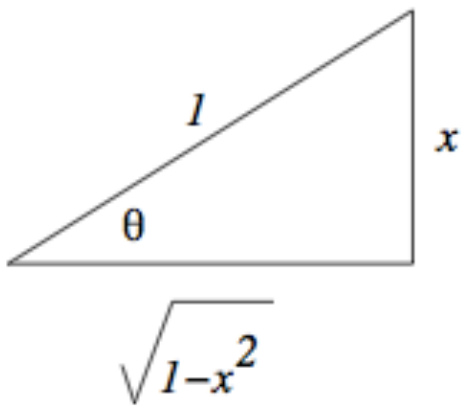
Inverse Trig Functions

- Let $f(x) = \sin(x)$ then $f^{-1}(x) = \arcsin(x)$

"the angle whose sine is x"

Meaning of Inverse Trig Functions

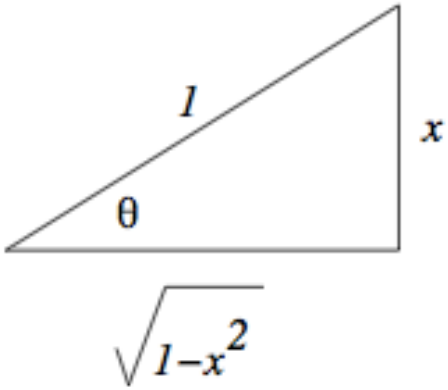
- Let $f(x) = \sin(x)$ then $f^{-1}(x) = \arcsin(x)$



"the angle whose sine is x"

- The above triangle is constructed so that $\sin(\theta)=x$, which means that $\theta = \arcsin(x)$

(1) Manipulating Inverse trig functions



Simplifying the expression

y=tan(arcsin(x)) leads to:

(A) $\sqrt{1-x^2}$

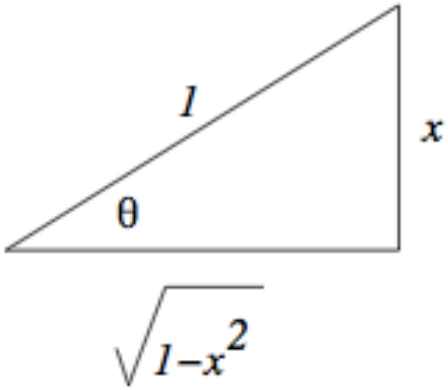
(B) $x/\sqrt{1-x^2}$

(C) $1/x$

(D) x

(E) $\sqrt{1-x^2}/x$

(1) Manipulating Inverse trig functions



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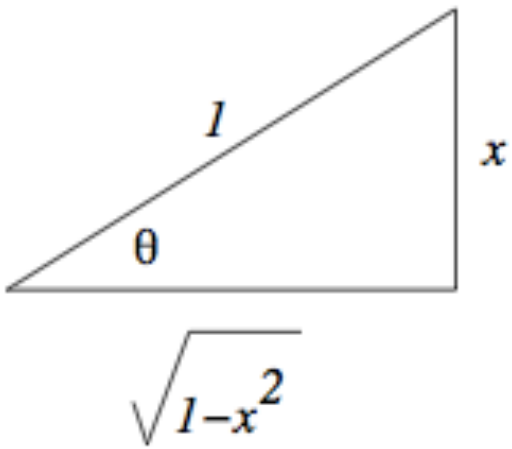
Manipulating Inverse trig functions

Simplifying the expression y=tan(arcsin(x)):



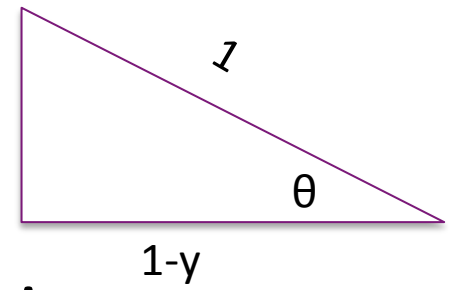
θ = arcsin(x) and tan(θ)=opp/adj = $x/\sqrt{1-x^2}$

so y=tan(arcsin(x)) = $x/\sqrt{1-x^2}$

Back to cycloid

$$x = \theta - \sin\theta,$$
$$y = 1 - \cos\theta,$$

- $y = 1 - \cos\theta, \quad \rightarrow \quad 1 - y = \cos\theta$



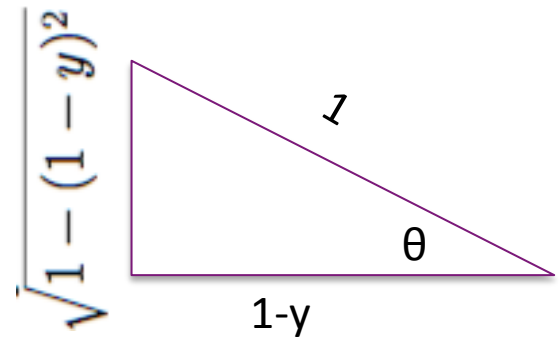
- Draw a right triangle with this relation satisfied. We can use that to find all other trig quantities
- Use the inverse trig fn: $\arccos(1-y) = \theta$
- Plug into the equation for x and simplify

$$x = \arccos(1-y) - \underline{\sin(\arccos(1-y))}$$

Back to cycloid

$$x = \theta - \sin\theta,$$
$$y = 1 - \cos\theta,$$

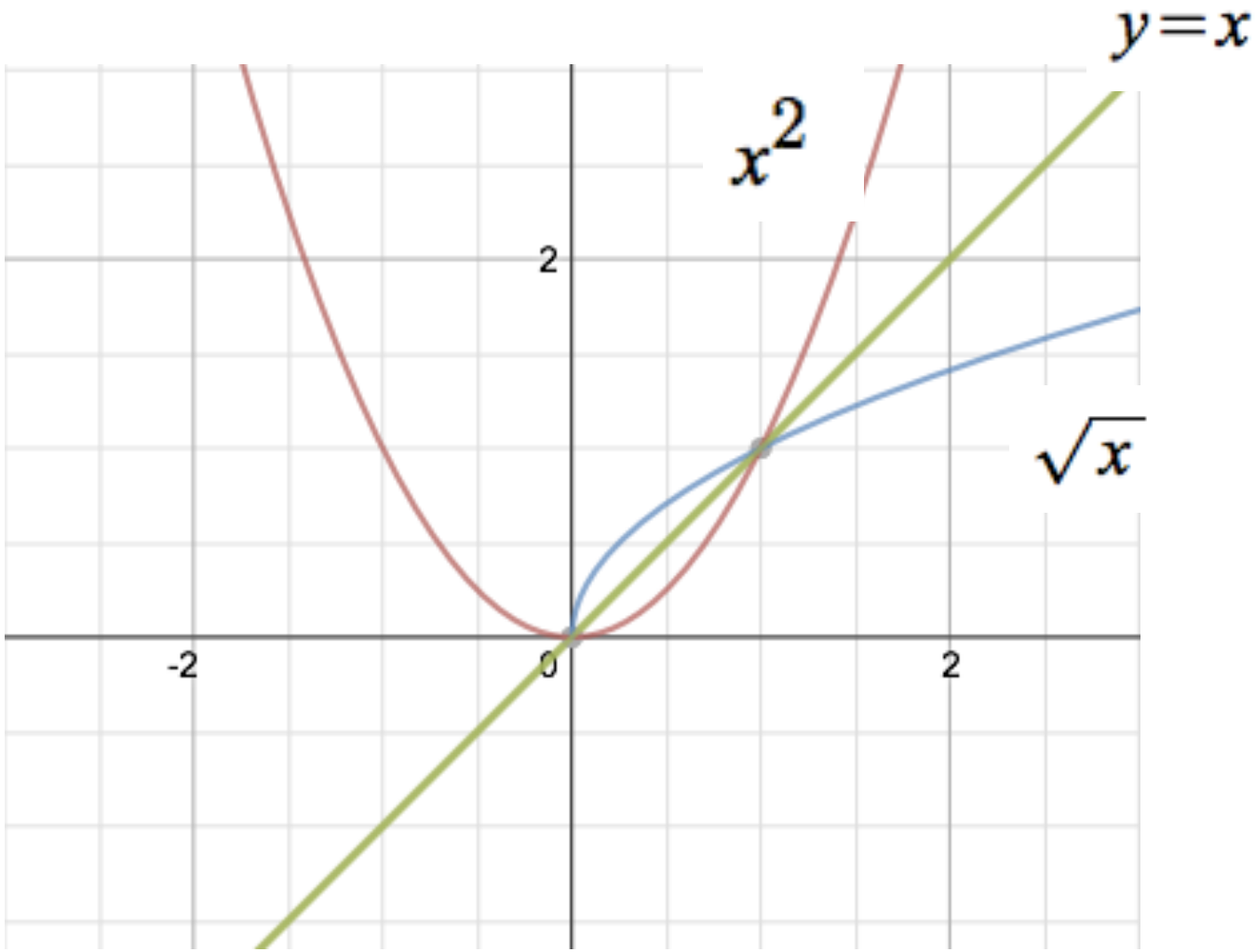
- Simplify by using this triangle
- Sin(θ) = opp/hypot



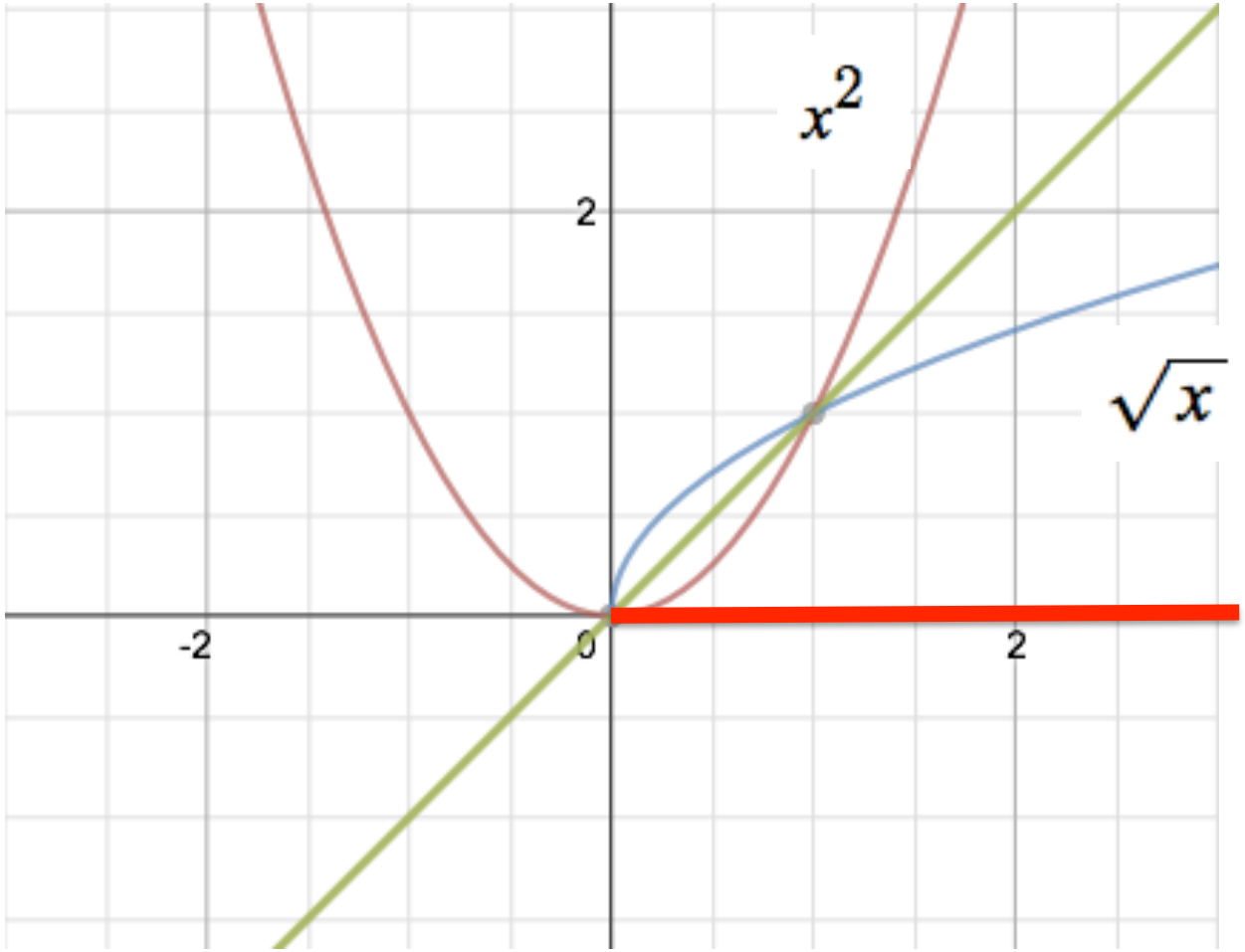
$$x = \arccos(1-y) - \sin(\arccos(1-y))$$

$$x = \arccos(1-y) - \sqrt{1-(1-y)^2}$$

Inverse functions

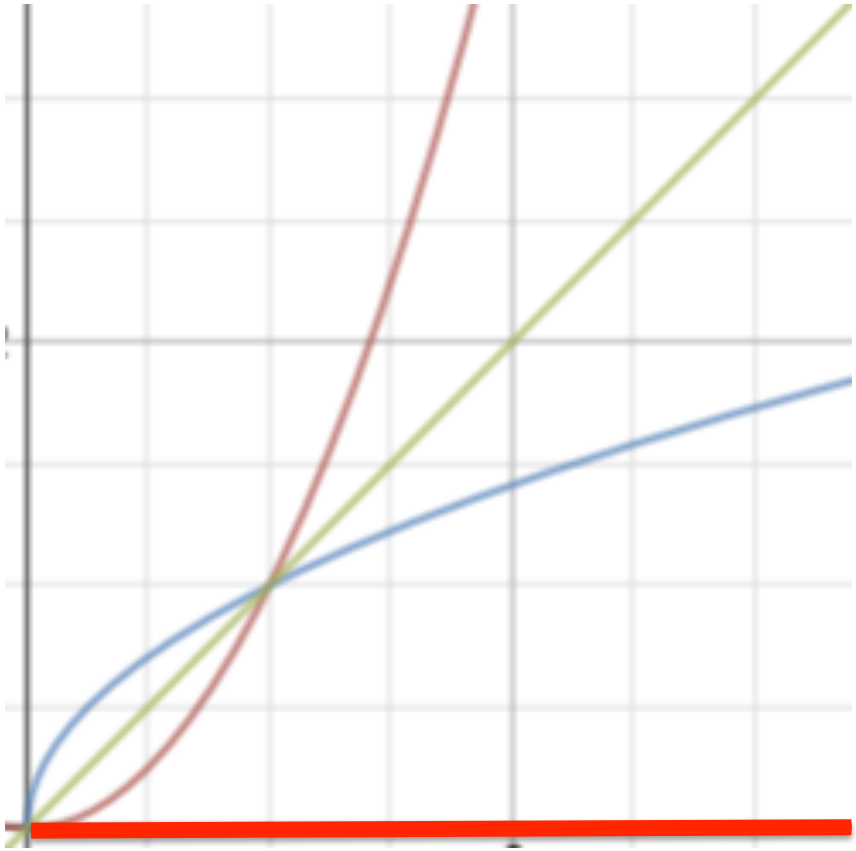


Inverse functions on restricted domains



On restricted domains

$f(g(x))=x$ and $g(f(x))=x$



Domains of Inverse Trig Functions

- Let $f(x) = \sin(x)$ then $f^{-1}(x) = \arcsin(x)$

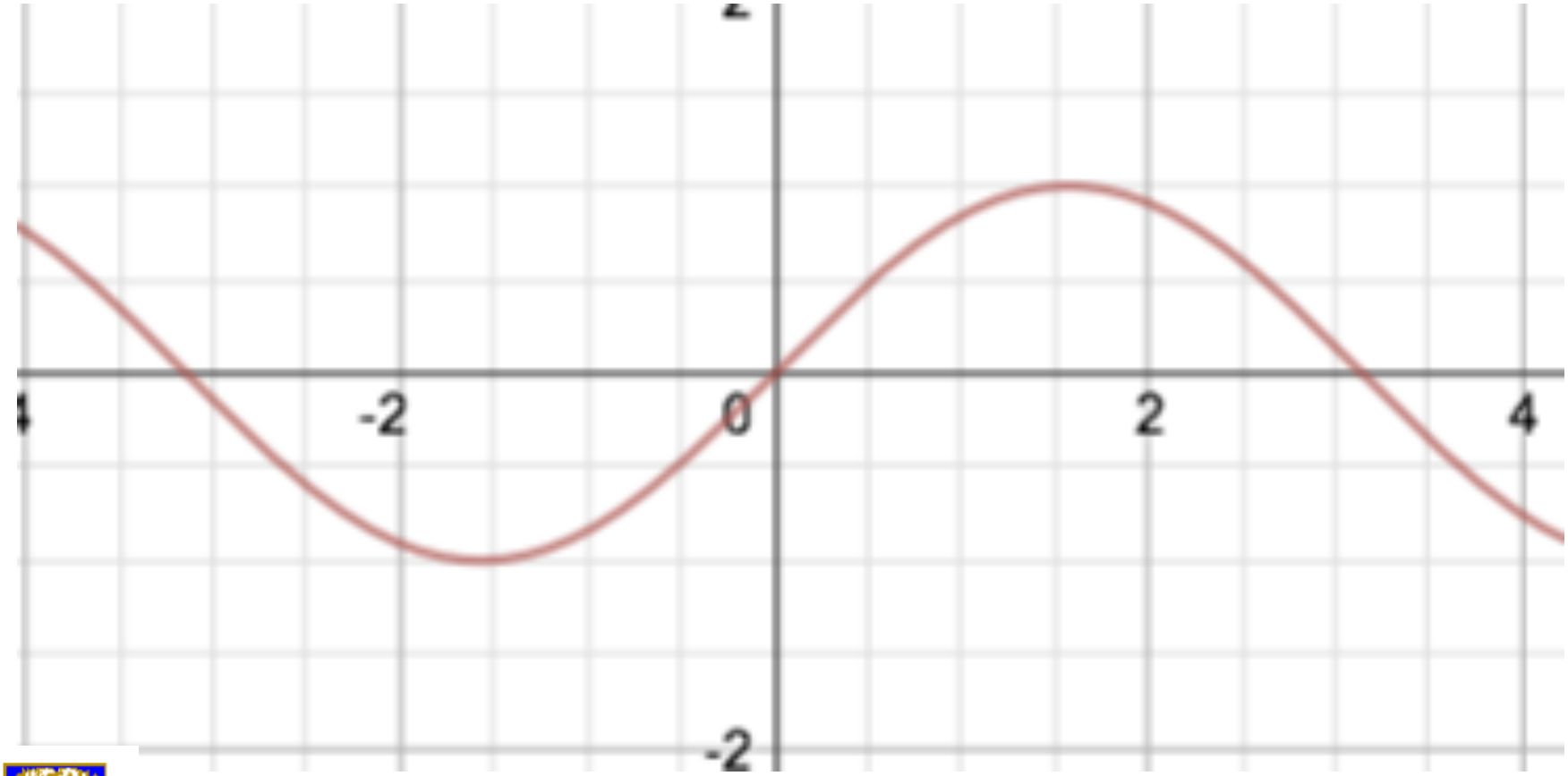
"the angle whose sine is x"

- Because $\sin(x)$ is periodic (repeats itself), we have to restrict the domain to define an inverse function.

Domain of sin(x)

1

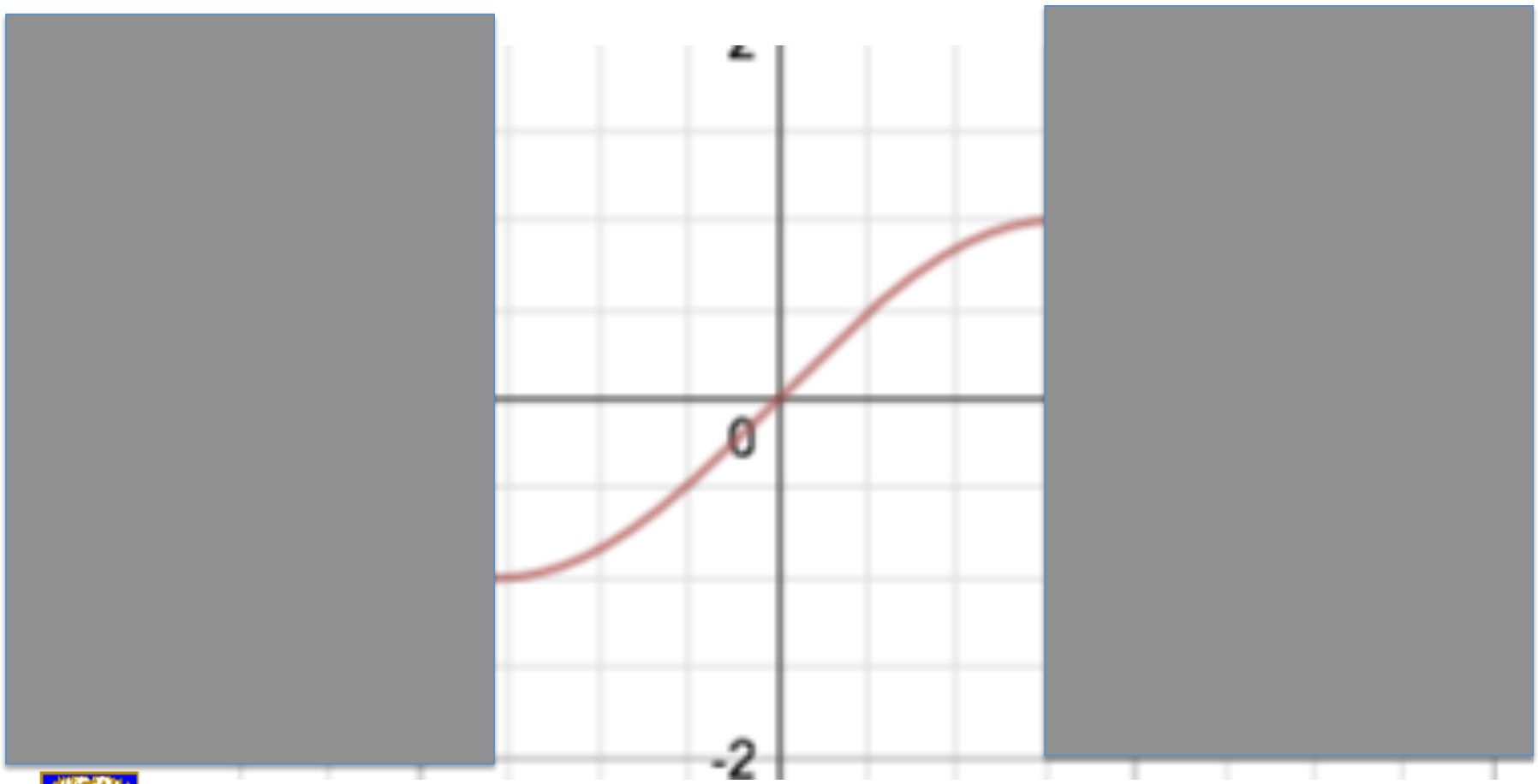
$$f(x) = \sin(x)$$



Restricting the domain

1

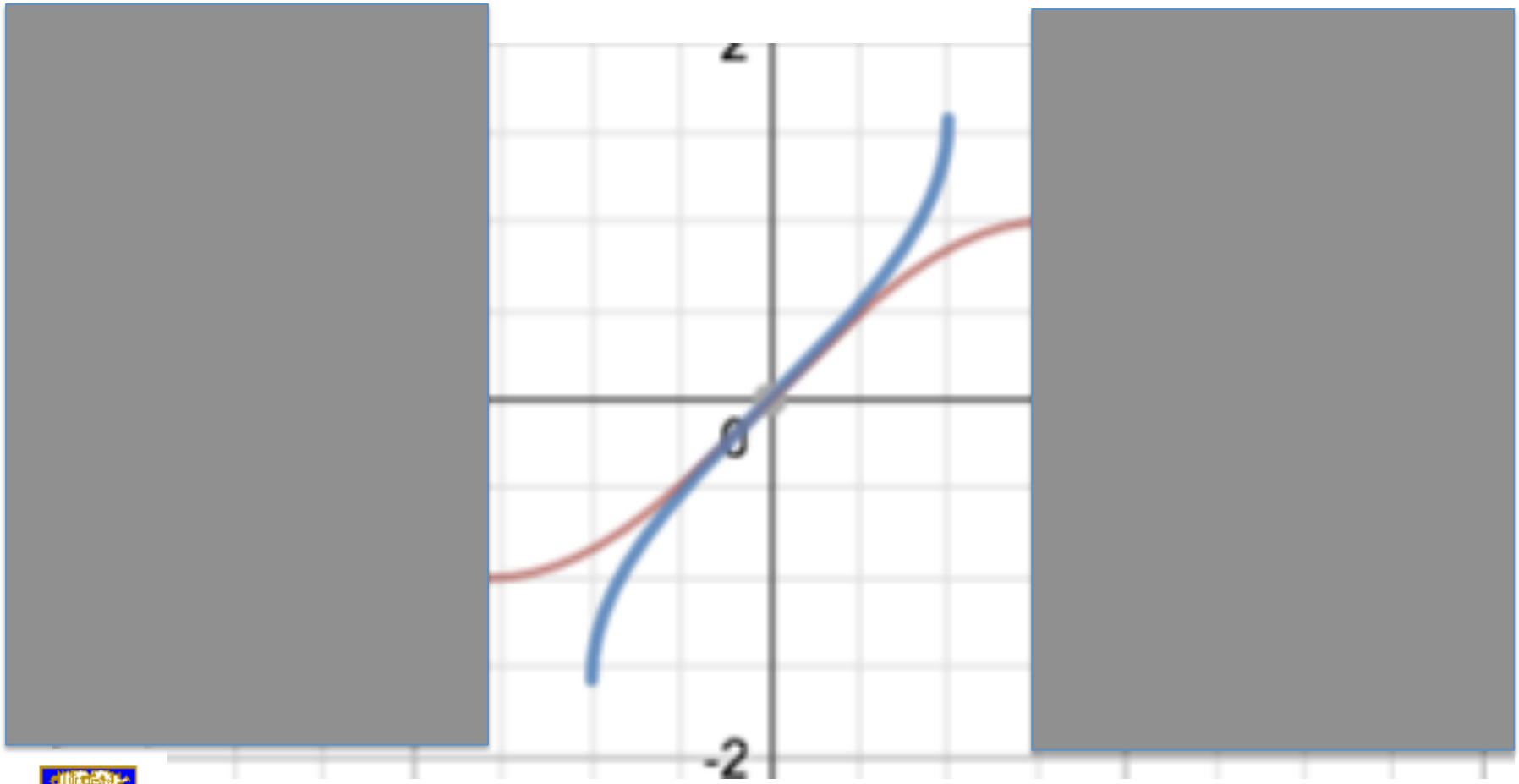
$$f(x) = \sin(x)$$



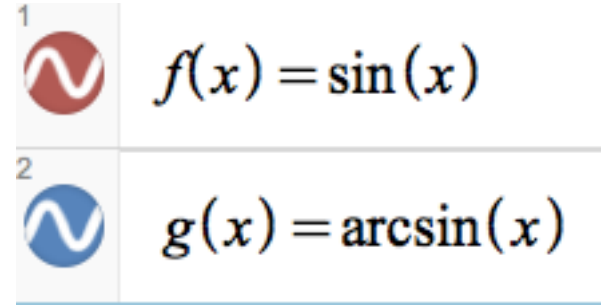
Domain of arcsin(x)

1 $f(x) = \sin(x)$

2 $g(x) = \arcsin(x)$



(2) Domains



The functions sin(x) and arcsin(x) are inverse functions on the following domains:

(A) $-\pi \le x \le \pi$ and $-1 \le x \le 1$

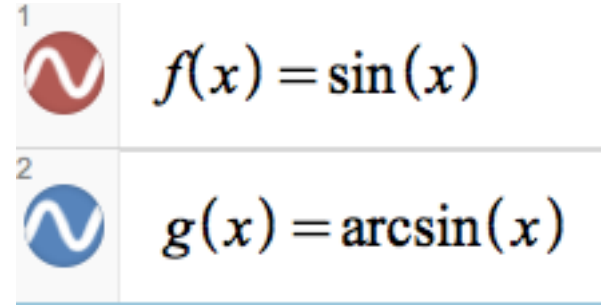
(B) $-\pi/2 \le x \le \pi/2$ and $-1 \le x \le 1$

(C) $-1 \le x \le 1$ and $-\pi/2 \le x \le \pi/2$

(D) $-1 \le x \le 1$ and $-\pi \le x \le \pi$

(E) $-\pi/2 \le x \le \pi/2$ and $-\pi/2 \le x \le \pi/2$

(2) Domains



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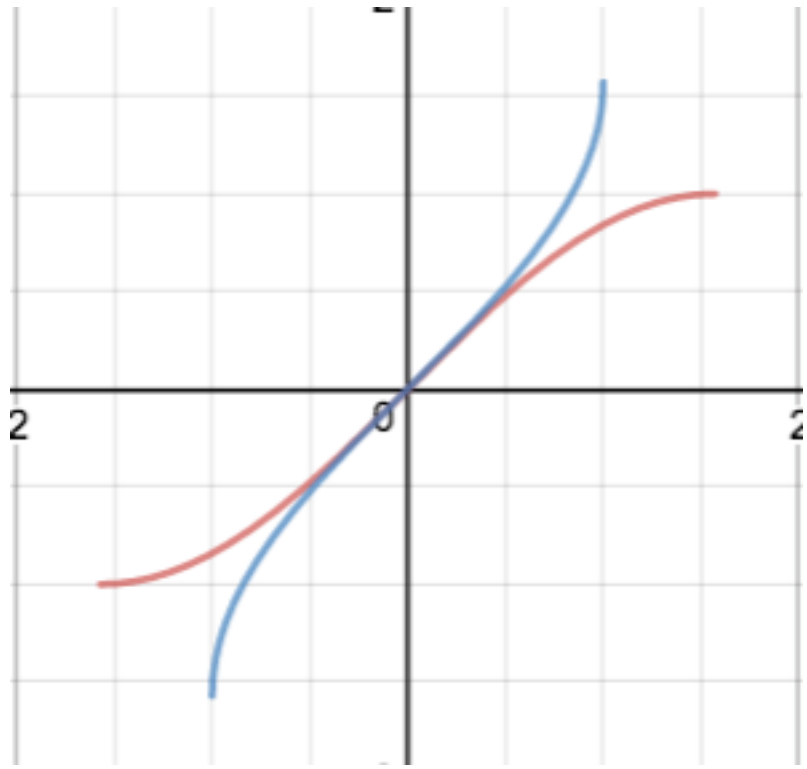
(E) $-\pi/2 \le x \le \pi/2$ and $-\pi/2 \le x \le \pi/2$

On the restricted domains,
arcsin(sin(x))=x

$\sin(x)\left\{-\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$

2 $\arcsin(x)$

•



$-\pi/2 \leq x \leq \pi/2$ and $-1 \leq x \leq 1$

Similarly, cosine and arccosine

- Domains

$\cos(x)\{0 < x < \pi\}$

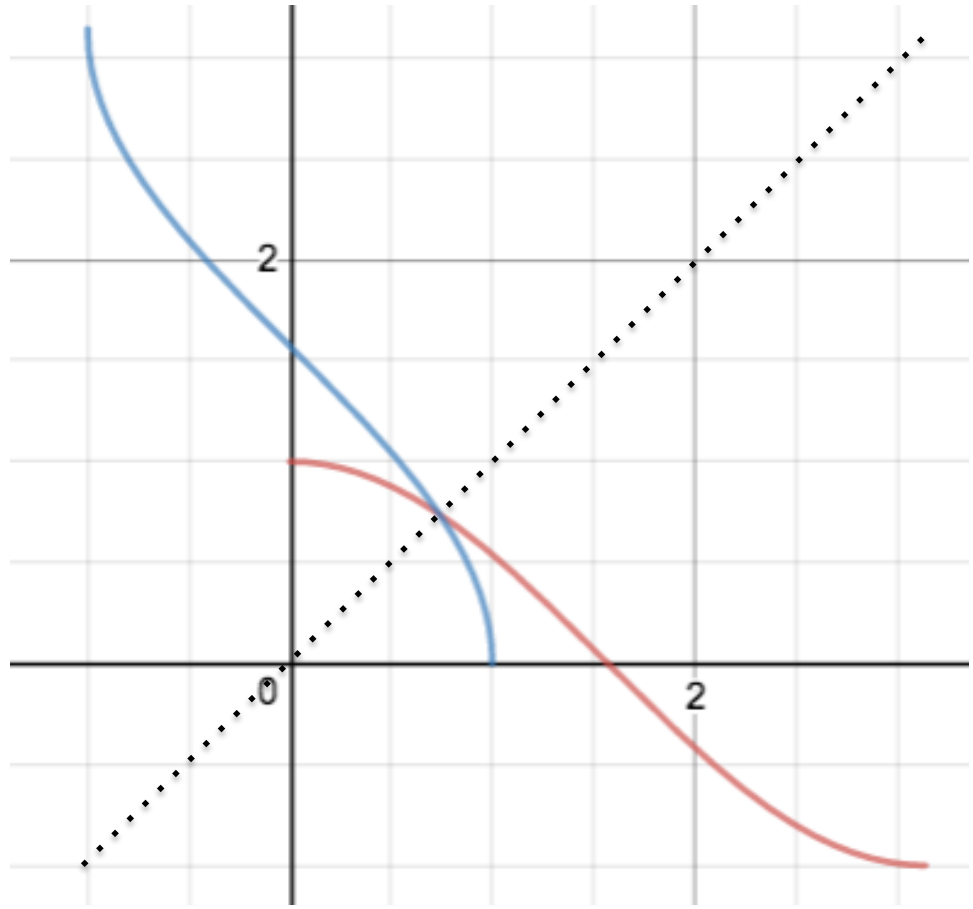
$\arccos(x)$

Symmetry about y=x

- Domains

$\cos(x)\{0 < x < \pi\}$

$\arccos(x)$



What is the derivative of arccos(x)?

- (A) – arcsin(x)
- (B) arccos (x)
- (C) $\sqrt{1-x^2}$
- (D) - $1/\sqrt{1-x^2}$
- (E) $1/(1+x^2)$

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Derivative of arccos(x)

- Rewrite in terms of familiar function

$$y = \arccos(x) \quad \Rightarrow \quad x = \cos(y)$$

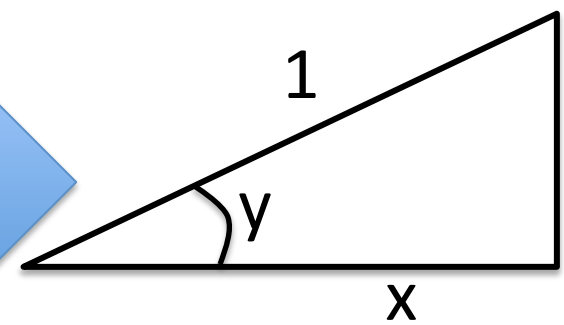
- Now use implicit differentiation

This side is just = 1

$$\frac{dx}{dx} = -\sin(y)\frac{dy}{dx}$$

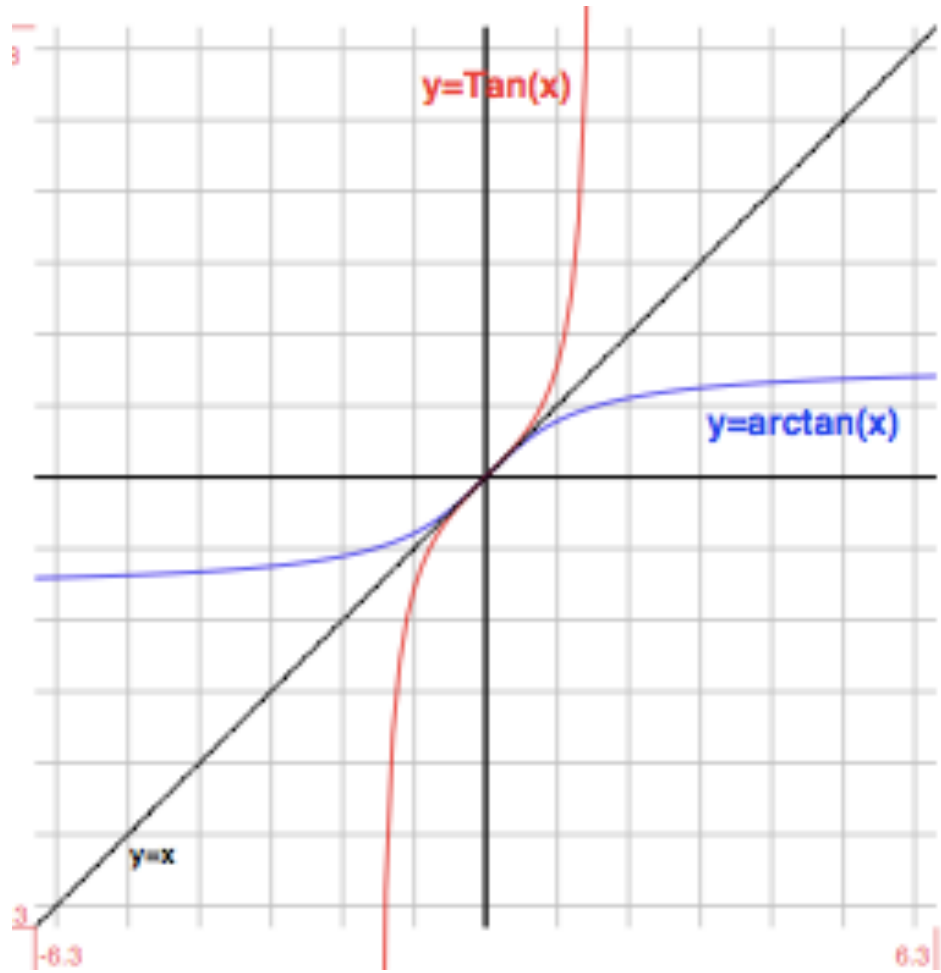
$$\frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{(1-\cos^2(y))^{1/2}} = -\frac{1}{\sqrt{1-x^2}}$$

A triangle in which x = cos(y). We can use it to express sin(y) in terms of x.



Tan(x) and arctan(x)

Figure out the domains of each of these functions



Derivatives of Tan(x) and arctan(x)

$$\boxed{\frac{d\tan(x)}{dx} = \sec^2(x)}$$

(Show this using quotient rule on $\tan(x) = \frac{\sin(x)}{\cos(x)}$)

$$\boxed{\frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}}$$

Rewrite as x=tan(y), use implicit differentiation

Most important derivatives of inverse trig functions

$y = f(x)$	$f'(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{x^2+1}.$

Practice of this and implicit diff.

- Find an equation for dy/dx for the curve

$$x = \arccos(1-y) - \sqrt{1-(1-y)^2}$$

- (WS Nov 27 Q 6)

$$\frac{dx}{dx} = \frac{d\arccos(1-y)}{dy}\frac{dy}{dx} - \frac{d\sqrt{1-(1-y)^2}}{dy}\frac{dy}{dx}$$

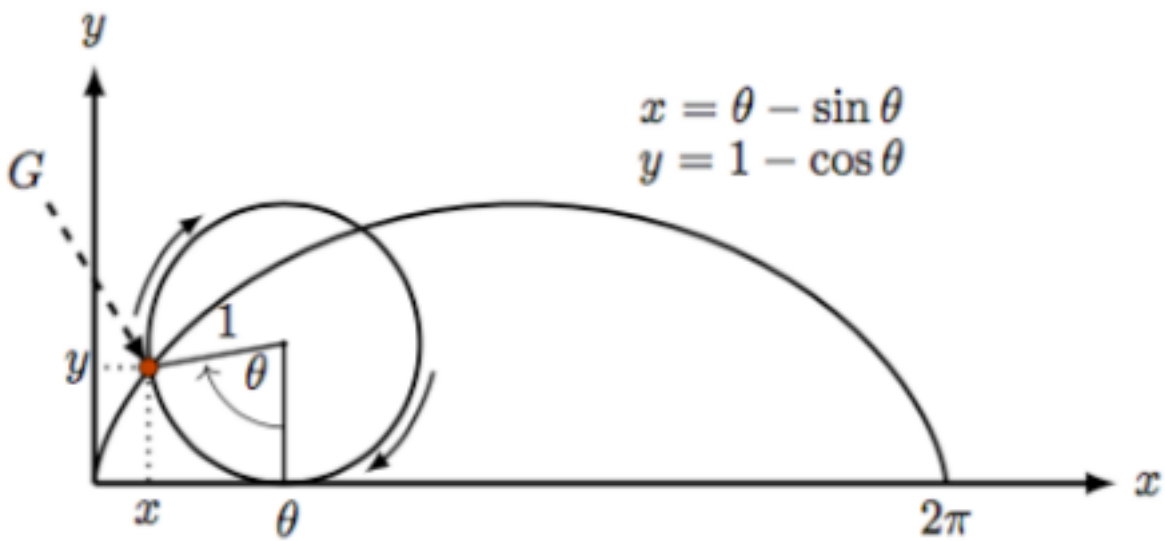
$$1 = \frac{-1}{\sqrt{1-(1-y)^2}} \cdot (-1) \cdot \frac{dy}{dx} - \frac{1}{2\sqrt{1-(1-y)^2}} \cdot -2(1-y) \cdot (-1) \cdot \frac{dy}{dx}$$

- Simplifies to:

$$\left[1 + [y'(x)]^2\right] y = \text{constant}$$

Cycloid

The cycloid.



- A living demo of the cycloid…

Cycloid on Desmos:
1.Prepare for animation

- Start by defining a parameter that will get animated later

- Give it a range of $0 < a < 6\pi$

2. Add a circle

- Input the equation of a circle of radius 1 centered at the origin.

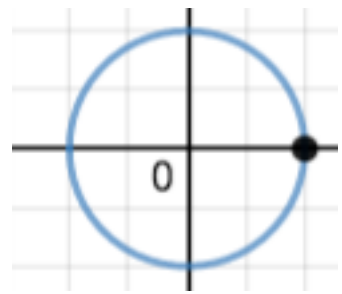
- Add a point on the rim of the circle using sine and cosine of *a* as the x and y coordinates.

- Make your point go around the circle by animating *a*

It should look like:

$$x^2 + y^2 = 1$$

$$\big(\cos(a), \sin(a)\big)$$



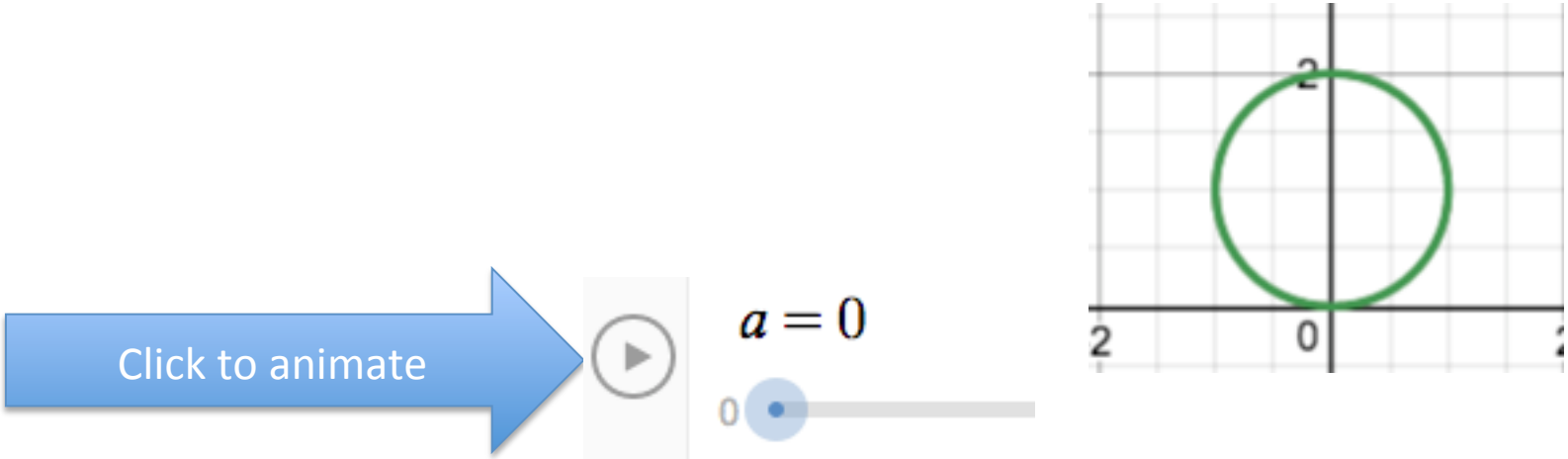
Animating *a* will make the point move around the circle.

OK, now turn off that point, so we can move on..

Make your circle move

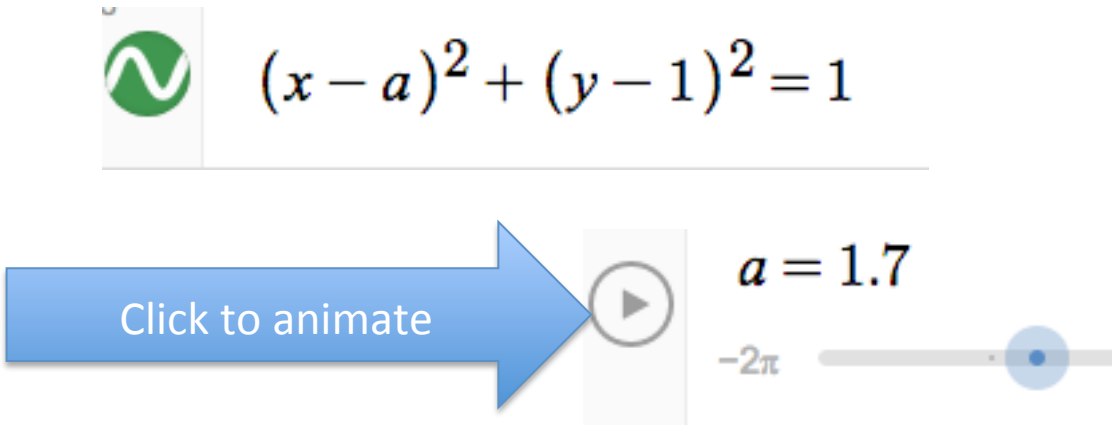
- Change the equation of the circle so that its center is at (a,1)

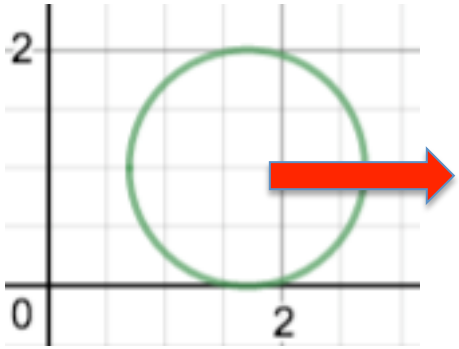
- What happens when you animate a?



It should look like:

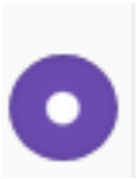
- You should see a circle sliding up the x axis





Add the coordinates of the cycloid

- Add this:

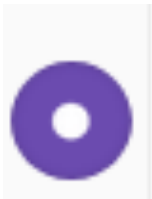


$$\left(a - \sin(a), 1 - \cos(a)\right)$$

- See what happens when you animate *a*

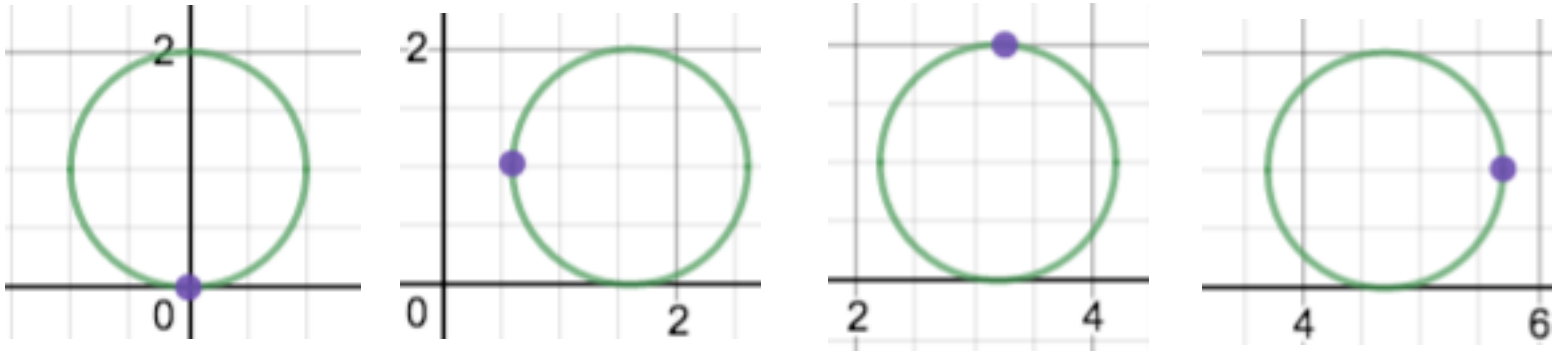
It should look like:

- A point stuck on the rim of the rolling circle



$$\left(a - \sin(a), 1 - \cos(a)\right)$$

$a = 0$ $a = 1.6$ $a = 3.2$ $a = 4.7$



Add the cycloid path

$$(t - \sin(t), 1 - \cos(t))$$

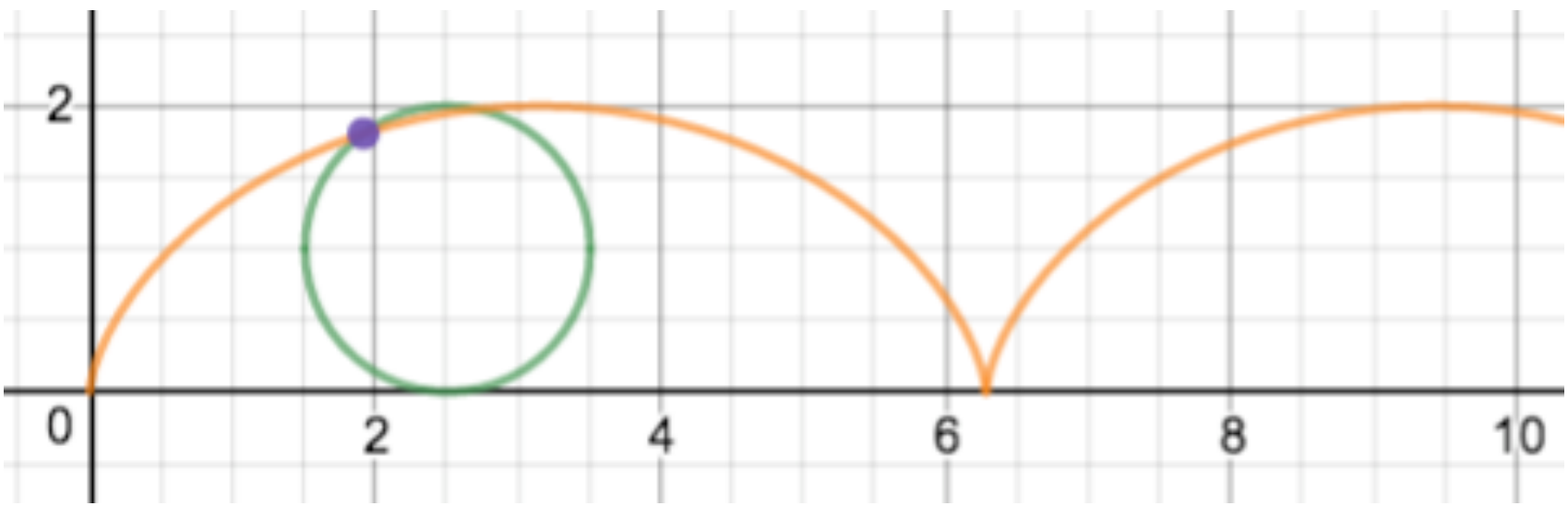
$$0 \leq t \leq 6\pi$$

- This will add the curve, not just the point already on your graph.

Add the cycloid path

- $(t - \sin(t), 1 - \cos(t))$

  $0 \leq t \leq 6\pi$

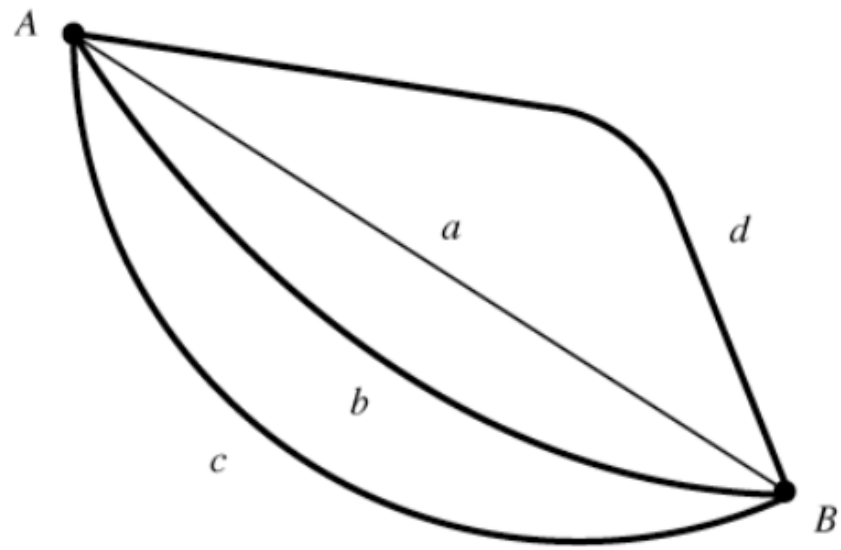


So who cares?

- What’s special about this funny curve?

Famous problem:

"Find the path of least time between points *A* and *B* for a particle moving under force of gravity."



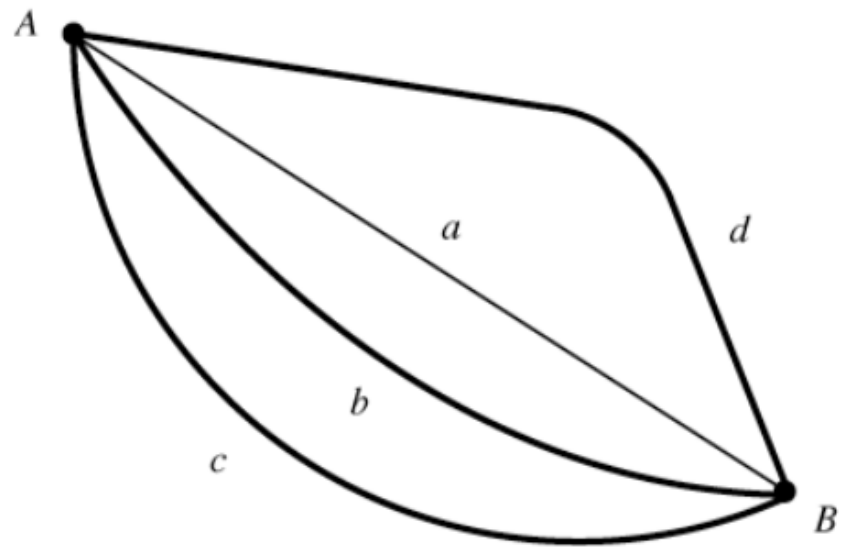
“Brachistochrone”

“Find the path of least time between points *A* and *B* for a particle moving under force of gravity.”

Johann Bernoulli 1667 – 1748

Posed the problem (1696)

https://en.wikipedia.org/

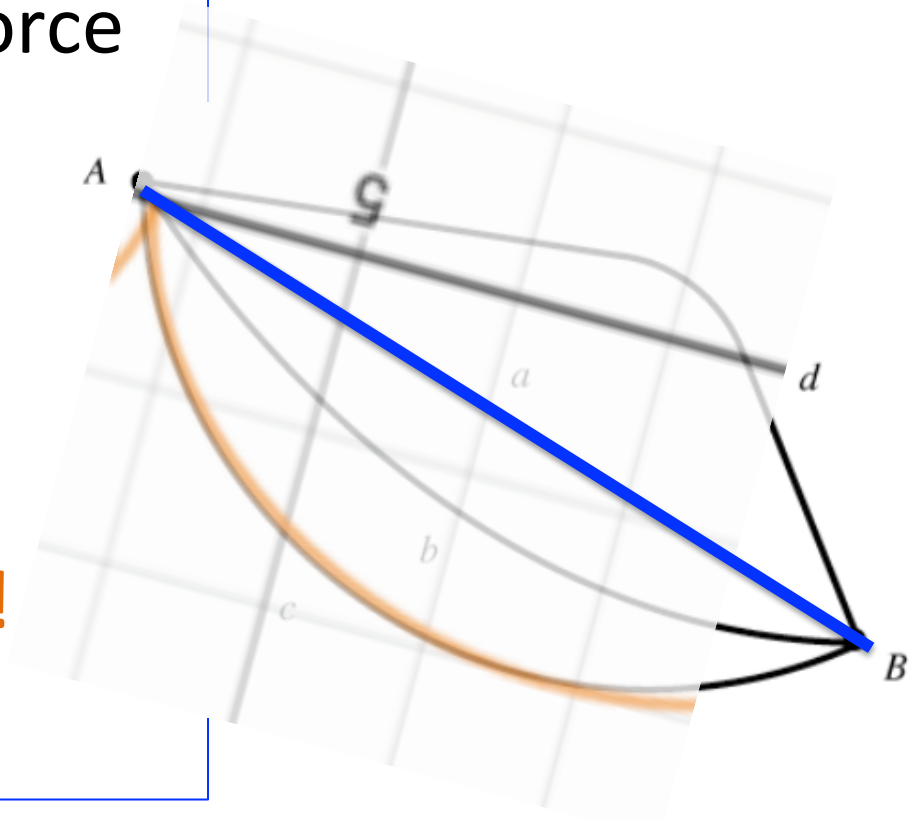


Solution is the cycloid!

It is the path of least time between points *A* and *B* for a particle moving under force of gravity.

Straight line = shortest distance

Cycloid = shortest time!!



Demonstration

- Marble race on a linear and cycloidal path.

A history of quarrels

Gilles de Roberval (1628)

I know the secret, but I'll never tell you the solution to this exam problem.

Pierre de Fermat

I solved it too!

Rene Descartes

The most ridiculous gibberish I've ever seen

https://en.wikipedia.org/wiki/Gilles_de_Roberval
https://en.wikipedia.org/wiki/Ren%C3%A9_Descartes

Tautochrone

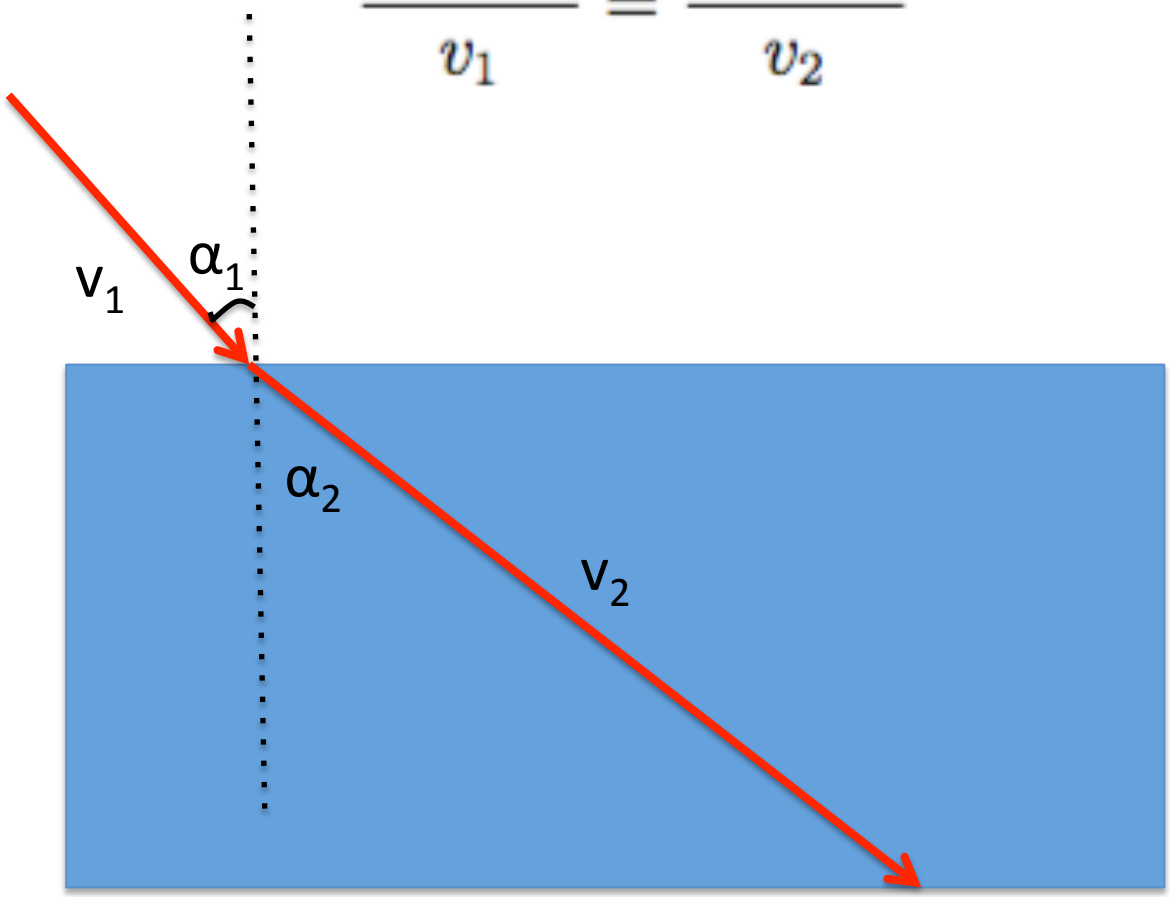
- Find a curve such that sliding down that to the lowest point does not depend on the starting point
- Solution: a cycloid, time = $\pi\sqrt{r/g}$
- Same curve as the brachistochrone!

- Solution by Christiaan Huygens 1659

https://en.wikipedia.org/wiki/Tautochrone_curve
https://en.wikipedia.org/wiki/Christiaan_Huygens

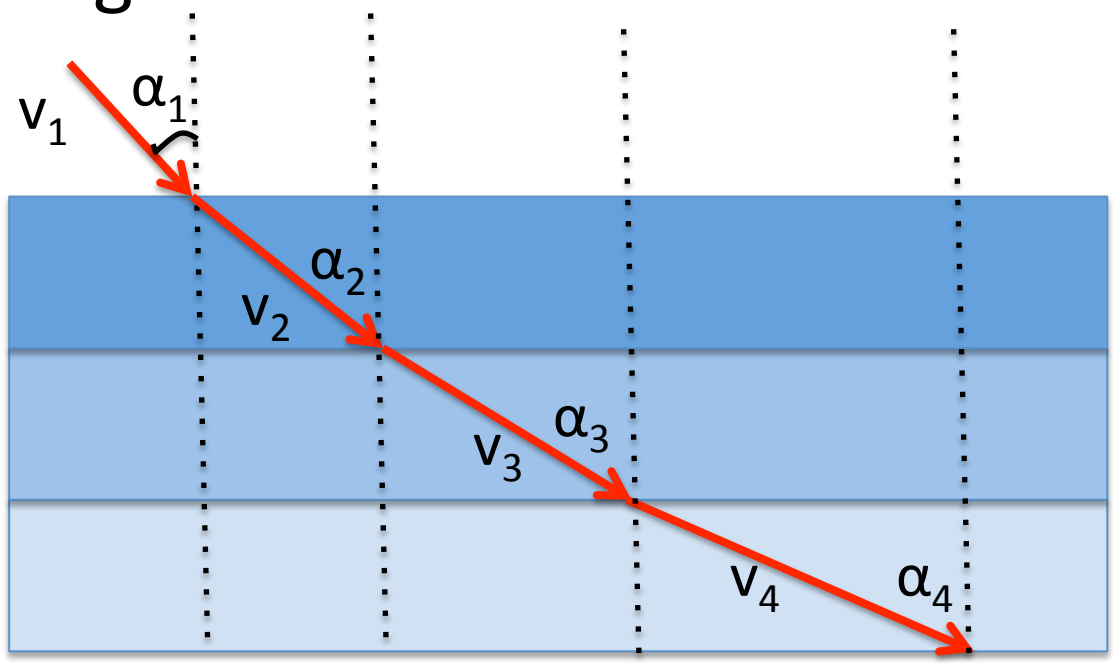
Johan Bernoulli's solution: uses Snell's law

$$\frac{\sin(\alpha_1)}{v_1} = \frac{\sin(\alpha_2)}{v_2}$$



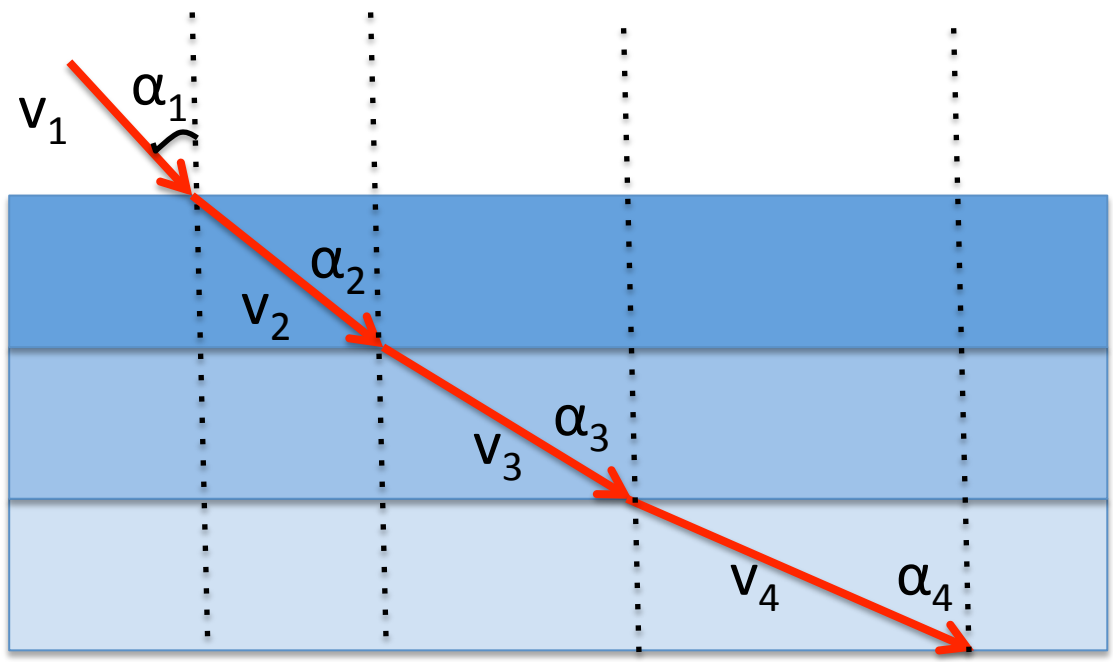
Suppose there are many layers

- Light passing through layers with decreasing density: speed increases, and light ray bends according to Snell's Law



Snell's law for many layers

$$\frac{\sin(\alpha_1)}{v_1} = \frac{\sin(\alpha_2)}{v_2} = \frac{\sin(\alpha_3)}{v_3} = \text{Constant}$$



Idea

- Find a solution by immitating the behaviour of light

$$\frac{\sin(\alpha)}{v} = \text{Constant}$$

- Use two things:

(1) Velocity of particle changes due to gravity

(2) We can relate sin(α) to the slope of the tangent line of the curve (see previous lecture)

(1) velocity at a given height

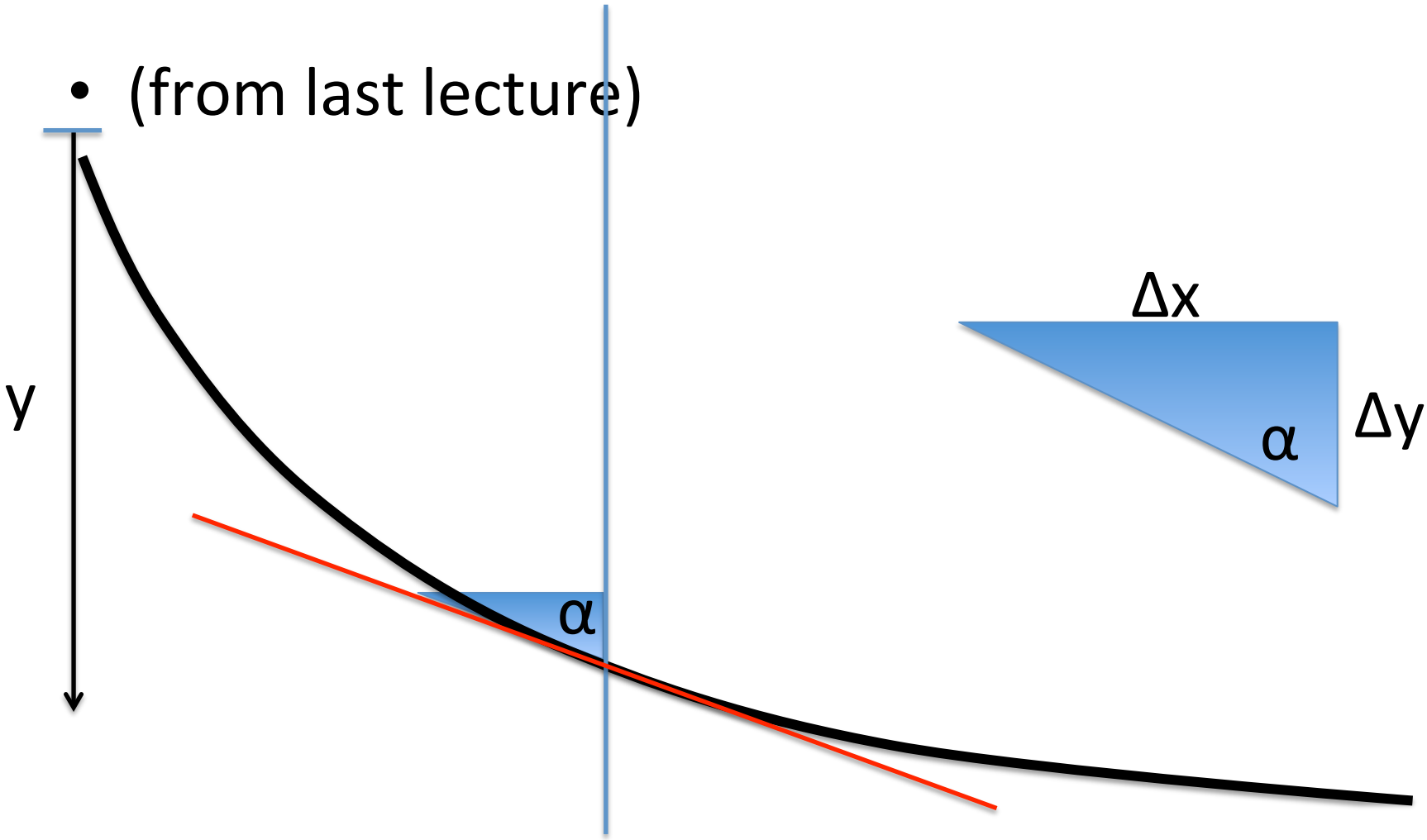
- Ball starts with velocity v=0 from some height

- After falling vertical distance y, it has traded its potential energy = m g y for kinetic energy:

$$\frac{1}{2} m v^2 = m g y \quad \rightarrow \quad \boxed{v = (2 g y)^{1/2}}$$

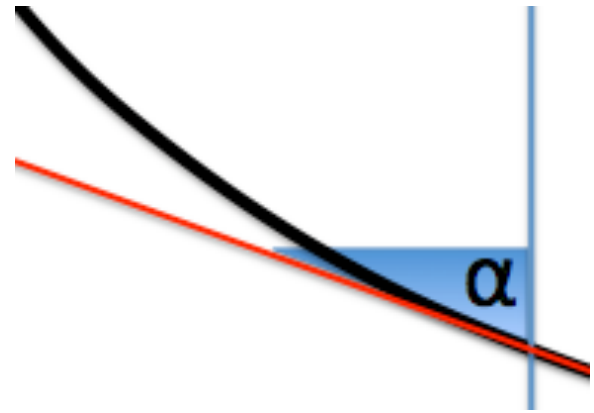
Relate sin(α) to dy/dx

- (from last lecture)

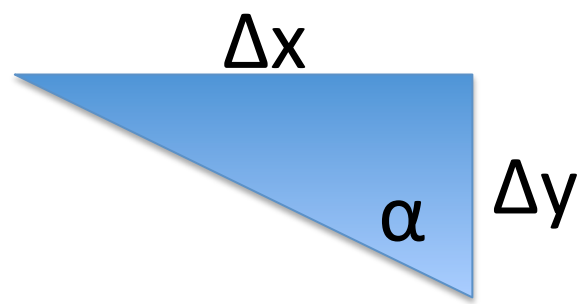


(2) sin(α)

We showed last week that we can relate this to the slope of the tangent line:



sin(α) = opp/hyp



$$\sin(\alpha) = \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\sin(\alpha) = \frac{1}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} = \frac{1}{\sqrt{1 + (y'(x))^2}}$$

Put these facts together

$$\frac{\sin(\alpha)}{v} = \text{constant}$$

$$\frac{1}{\sqrt{1+(y'(x))^2}} \cdot \frac{1}{\sqrt{2gy}} = A$$

$$\sqrt{1+(y'(x))^2} \cdot \sqrt{2gy} = \frac{1}{A}$$

$$\left(1+(y'(x))^2\right) \cdot 2gy = \frac{1}{A^2}$$

$$\left(1+(y'(x))^2\right) \cdot y = \text{constant}$$

What it means:

Conclusion: The differential equation below describes the path of least time:

$$\left(1 + (y'(x))^2\right) \cdot y = \text{constant}$$

We can show that the cycloid satisfies this differential equation using either implicit differentiation or trig identities!

(Calculation shown in the Appendix: document camera work)

JOHANNIS BERNOULLI,

M. D. MATHESEOS PROFESSORIS,
Regiarum Societatum PARISIENSIS, LONDINENSIS, PETROPOLITANÆ, BEROLINENSIS, *Socii* &c.

OPERA OMNIA,

TAM ANTEA SPARSIM EDITA, quam hactenus inedita.

TOMUS PRIMUS,

Quo continentur ea
Quæ ab ANNO 1690 ad ANNUM 1713 prodierunt.

LAUSANNÆ & GENEVÆ,
Sumptibus MARCI-MICHAELIS BOUSQUET & Sociorum.

MDCCXLII.

Cum Privilegio Sacræ Cæsareæ Majestatis, & Serenissimi Poloniæ Regis, Electi. Saxon.

Johan Bernoulli's Collected work

- "Supra invidiam" (Above envy)
- https://www.maa.org/press/periodicals/convergence/mathematical-treasure-collected-works-of-johann-bernoulli

Cycloid: Helen of Geometry

- Beautiful properties but caused terrible quarrels between its mathematician “lovers” in 17th century.

The Helen of Geometry, John Martin

doi:10.4169/074683410X475083

Helen of Troy

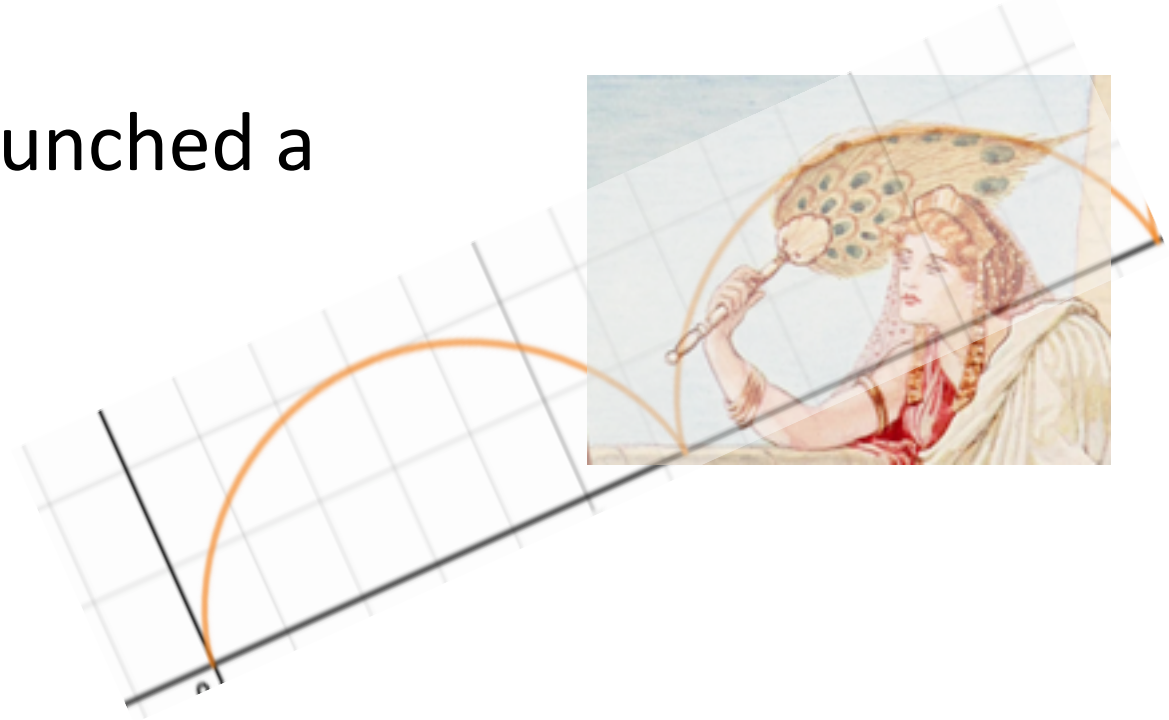
The face that launched a thousand ships

Lithograph by Walter Crane

https://en.wikipedia.org/wiki/Helen_of_Troy

Helen of Mathematics

The shape that launched a thousand quips.



Fill in a course evaluation

- https://eval.ctlt.ubc.ca/science

- Your constructive feedback and indication of which aspects of this course worked for you would be greatly appreciated.

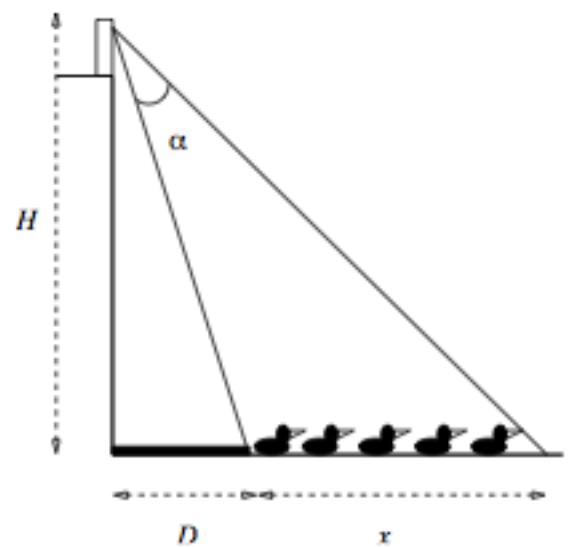
Problem solving session

We ran out of time, but showed the problem that we will solve next time.

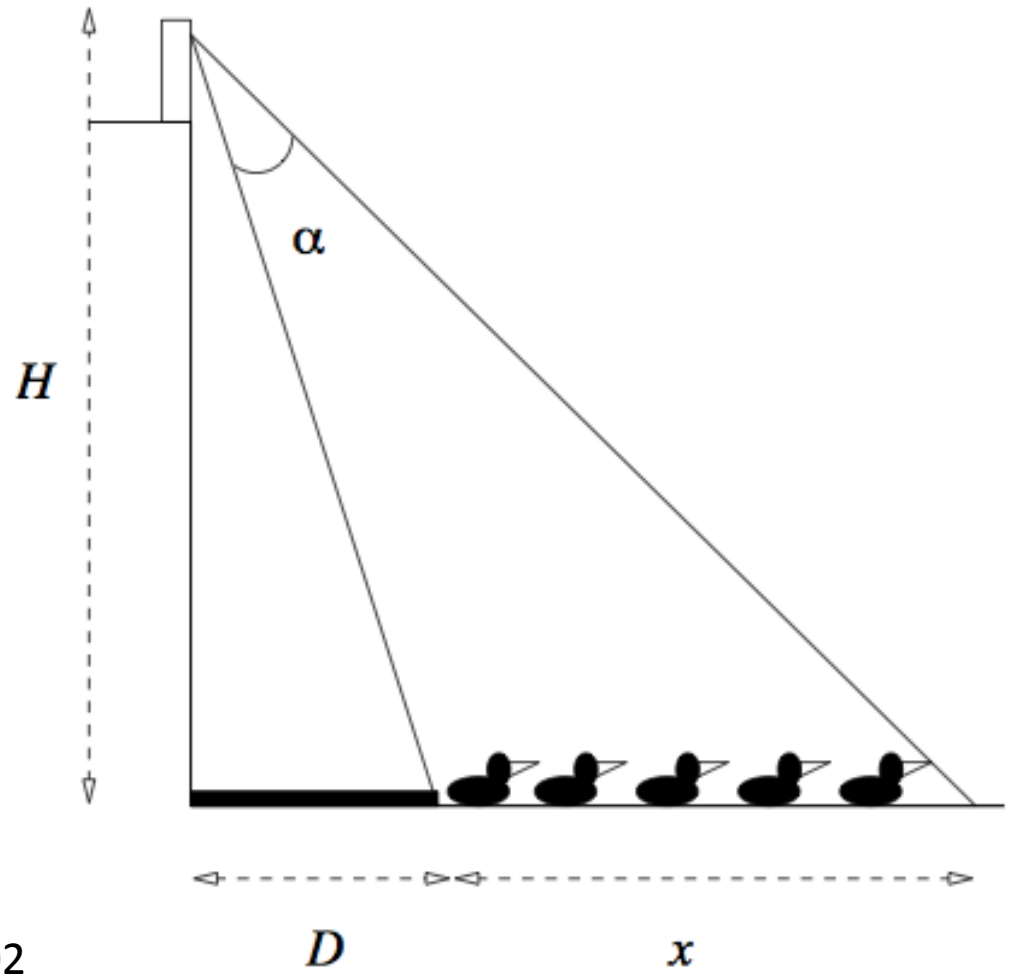
From recent research

Graduate student Ryan Lukeman studies behaviour of duck flocks swimming near Canada Place in Vancouver, BC. This figure from his PhD thesis shows his photography set-up. Here $H = 10$ meters is the height from sea level up to his camera aperture at the observation point, $D = 2$ meters is the width of a pier (a stationary platform whose size is fixed), and x is the distance from the pier to the leading duck in the flock (in meters). α is a visual angle subtended at the camera, as shown. If the visual angle is increasing at the rate of 1/100 radians per second, at what rate is the distance x changing at the instant that $x = 3$ meters?

See next slide for enlarged diagram



Enlarged diagram



Other practice questions:

- See our Question Challenge wiki

http://wiki.ubc.ca/Course:MATH102/Question_Challenge