# Trigonometric functions

And revolutions around a circle

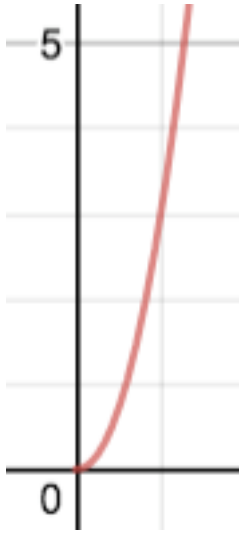
# Pre-exam/review/study

- Dec 12:  morning 10AM-12
- Dec 12 afternoon  1 – 2 PM

Location: Math Annex 1111

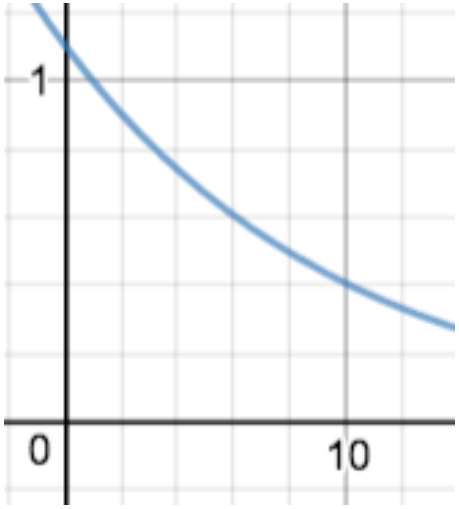
There will be similar sessoins with other instructors TBA

# CSP workshop: Some common misconceptions




- $m(t) = A\,t^{B}$ is not an exponential function!

- For an exponential function: the half life and decay rate are always related by $k = \ln(2)/\text{halflife}$
- (This does not depend on when the process was started, and whether the graph is shifted by 1 day or not).



# Is cricket farming a real deal?

Aspire Food Group

(Winners of the 2013 Hult Prize)

https://www.youtube.com/watch?v=klqWkFr8ei0

https://www.youtube.com/watch?v=eOLFu9sDl7o

# Hult prize 2013

# Total mass (to be optimized):

$$M(t) = m(t)N(t)$$
$$= aN_0 t^b e^{-kt}$$

Largest mass: find time at which M'(t)=0

# Best practice

- Consider first working with tidy formula

$$M'(t) = aN_0\left(bt^{b-1}e^{-kt} - kt^b e^{-kt}\right)$$
$$= aN_0 e^{-kt}\left(bt^{b-1} - kt^b\right)$$

- Before plugging in your numbers

# Critical point (best harvest time)

0 = M'(t) $= aN_0e^{-kt}\left(bt^{b-1} - kt^b\right)$

$$bt^{b-1} - kt^b = 0$$

$$t^{b-1}(b - kt) = 0$$

$$\boxed{t = b/k} = b\tau/\ln(2)$$

Where $\tau$ is the half-life of the cricket population.

2002 December Q2(iv)

# Critical points

- Critical points of $y = x^3 e^{-x}$ occur at:

  (a) $x = 0, \pm 3$

  (b) $x = 0, \pm 1/3$

  (c) $x = 0, 1/3$

  (d) $x = 0, 3$

  (e) none of the above

2002 December Q2(iv)

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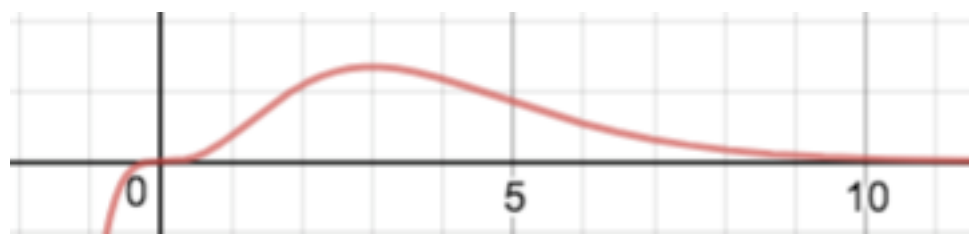
$x^3 e^{-x}$



# Quiz 3

Quiz grades:

Score=0.8(individual) + (2/5) group

# Q1 (based on WW problem)

Consider a cubic crystal of side length $x$ and volume $V = x^3$. If the volume is increasing at the rate of 3 $\text{mm}^3$ per month, and the crystal remains a cube, at what rate is its side length changing (in mm/month) when its volume is 8 $\text{mm}^3$?

$$\text{(A) } 3^{1/3}, \quad \text{(B) } \frac{3^{1/3}}{8}, \quad \text{(C) } \frac{9}{64}, \quad \text{(D) } \frac{1}{64}, \quad \text{(E) } \frac{1}{4}.$$

Individual


Group

# Q1 sol

- Related Rates

side length $x$ and volume $V = x^3$.

$$\frac{dV}{dt} = \frac{d}{dt}[x(t)^3] = 3x^2\frac{dx}{dt}$$

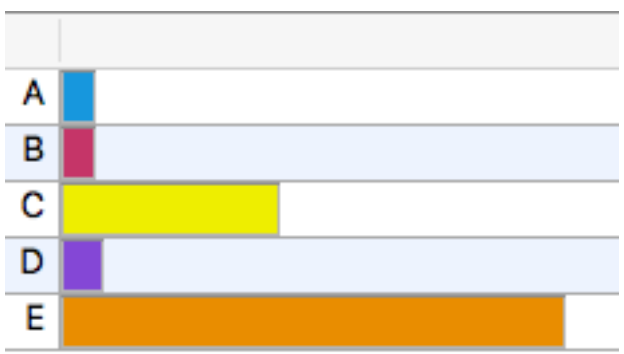
$$\frac{dx}{dt} = \frac{1}{3x^2}\frac{dV}{dt} = \frac{1}{3 \cdot 2^2} \cdot 3 = \frac{1}{4}$$

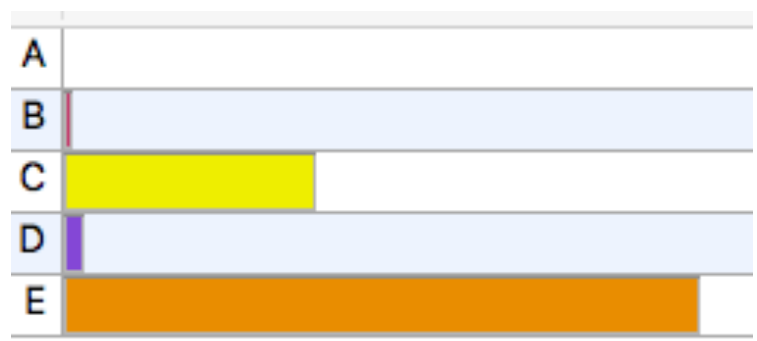
# Q2

The human population on Earth doubles roughly once every 50 years. If there are 7.5 billion people on Earth now, in how many years will there be 30 billion?

(A) $7.5e^{50t}$ years (B) $7.5e^{t/50}$ years (C) $\frac{7.5}{30}e^{t\ln(2)/50}$ years

(D) 75 years (E) 100 years



# Q2 sol

The current population is 7.5 billion. When the population doubles once, there would be 15 billion. When it doubles twice, it will be 30 billion. Hence it should take two doubling times, i.e. 100 years. The correct answer is (E).

# Q3

Consider the function $y = ax^b$. When I graph $\ln(y)$ versus $\ln(x)$, I see a line with slope 3/4 and vertical intercept 1. This tells me that

(A) $a = 1, b = 3/4$ (B) $a = 3/4, b = 1$ (C) $a = e, b = 3/4,$

(D) $a = 3/4, b = e$ (E) $a = 1, b = e^{\frac{3}{4}}$.

# Q3 sol

- Relationship: $y = ax^b.$

$$\ln(y) = \ln(ax^b)$$

$$\ln(y) = \ln(a) + \ln(x^b) = \ln(a) + b\ln(x).$$

- So: slope is b and intercept is ln(a), and hence

$$\ln(a) = 1 \qquad a = e^1 = e \qquad b = 3/4$$

# Q4

Which function is a solution to the differential equation $\frac{dh}{dt} = -k\sqrt{h}$ with $h(0) = h_0$?

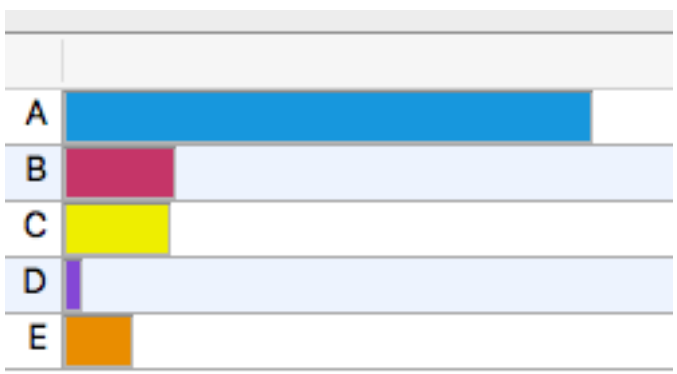
(A) $h(t) = (\sqrt{h_0} - \frac{1}{2}kt)^2$

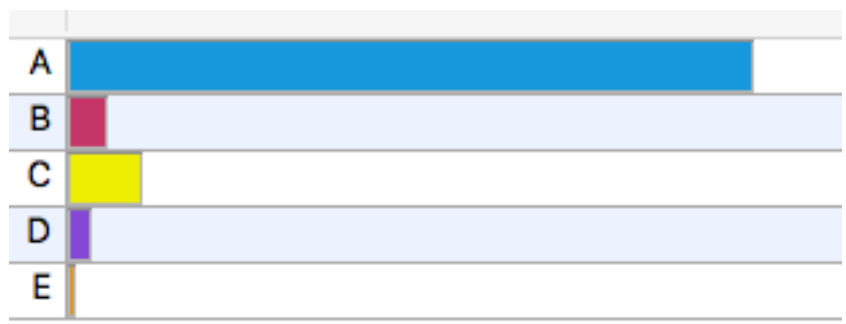
(B) $h(t) = (\sqrt{h_0} - kt^2)$

(C) $h(t) = h_0 e^{-kt}$

(D) $h(t) = (1 - e^{-kt})$

(E) None of the above



# Q4 Sol

- Two of the choices (B, D) do not satisfy the initial condition.
- Choice (C) satisfies a different DE (the one we studied for exponential growth and decay)
- In class before the quiz, we reviewed that the solution is

$$h(t) = (\sqrt{h_0} - \frac{1}{2}kt)^2$$

And this can be checked by differentiation.

# Q5

Which of the following differential equations has unstable steady states at $y = \pm 1$ and a stable steady state at $y = 0$? (Hint: consider sketching the state-space diagrams).

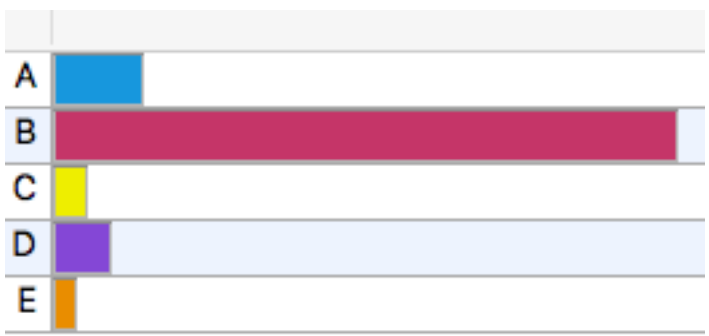
(A) $\dfrac{dy}{dt} = -2(y^3 - y)$

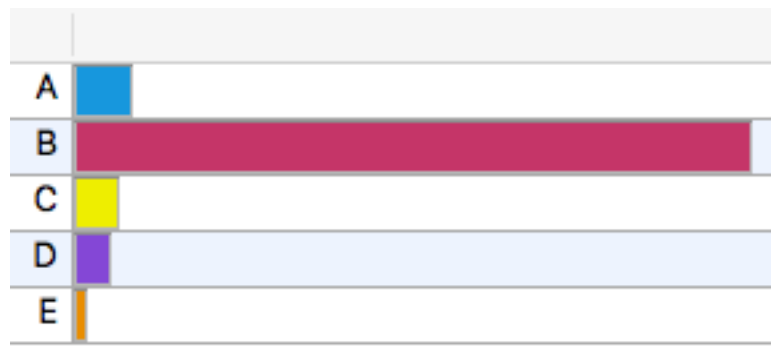
(B) $\dfrac{dy}{dt} = y^3 - y$

(C) $\dfrac{dy}{dt} = y(1 - y)^2$

(D) $\dfrac{dy}{dt} = y^4 - y^2$

(E) None of the above



# Q5 Sol

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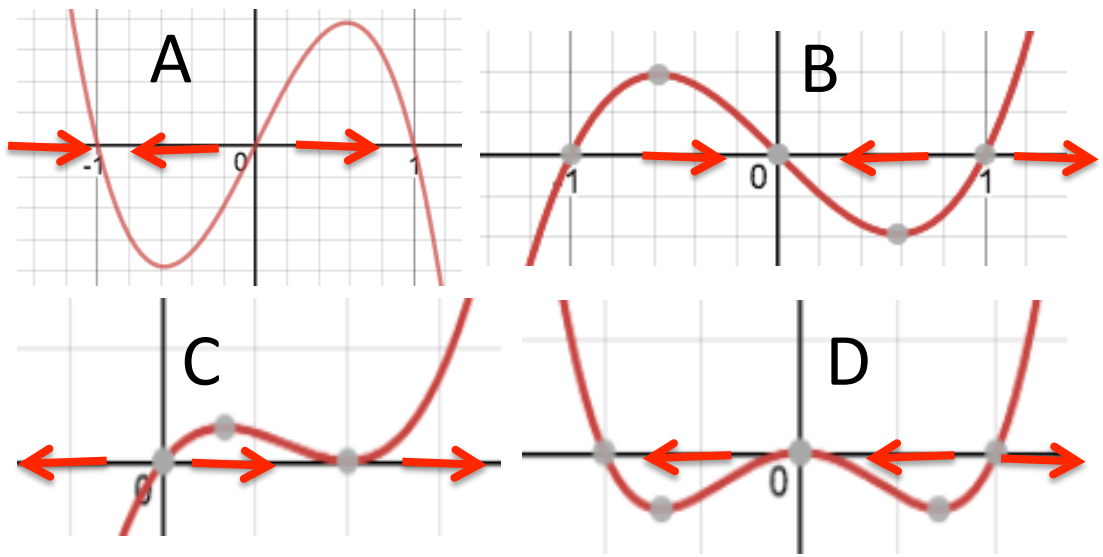


# Group version:
# Data from experiment in class

| Fluid height (cm) | Time (sec) |
|---|---|
| 14 | 0 |
| 12 | 13.46 |
| 10 | 26.33 |
| 8 | 40.86 |
| 6 | 58.35 |
| 4 | 79 |
| 2 | 106 |

# The behaviour should fit:

$$h(t) = (\sqrt{h_0} - \frac{1}{2}kt)^2$$

Or, if we transform the variables (sqrt both sides):

$$\sqrt{h} = \sqrt{h_0} - \frac{1}{2}kt$$

- So consider plotting sqrt(h) vs t, which should be a straight line with slope k/2 and intercept $\sqrt{h_0}$

# Emptying time:

$$h(t) = (\sqrt{h_0} - \frac{1}{2}kt)^2$$

- Set h=0 and find time t

$$t = 2\sqrt{h_0}/k$$

# At the end you have to transform back!

$$\sqrt{h} = \sqrt{h_0} - \frac{1}{2}kt$$

- sqrt(h) vs t is a straight line with slope -k/2 and intercept $\sqrt{h_0}$
- So hence k = -2 ( slope of line )
- $H_0$ = (intercept)$^2$

# Spreadsheet

- Data

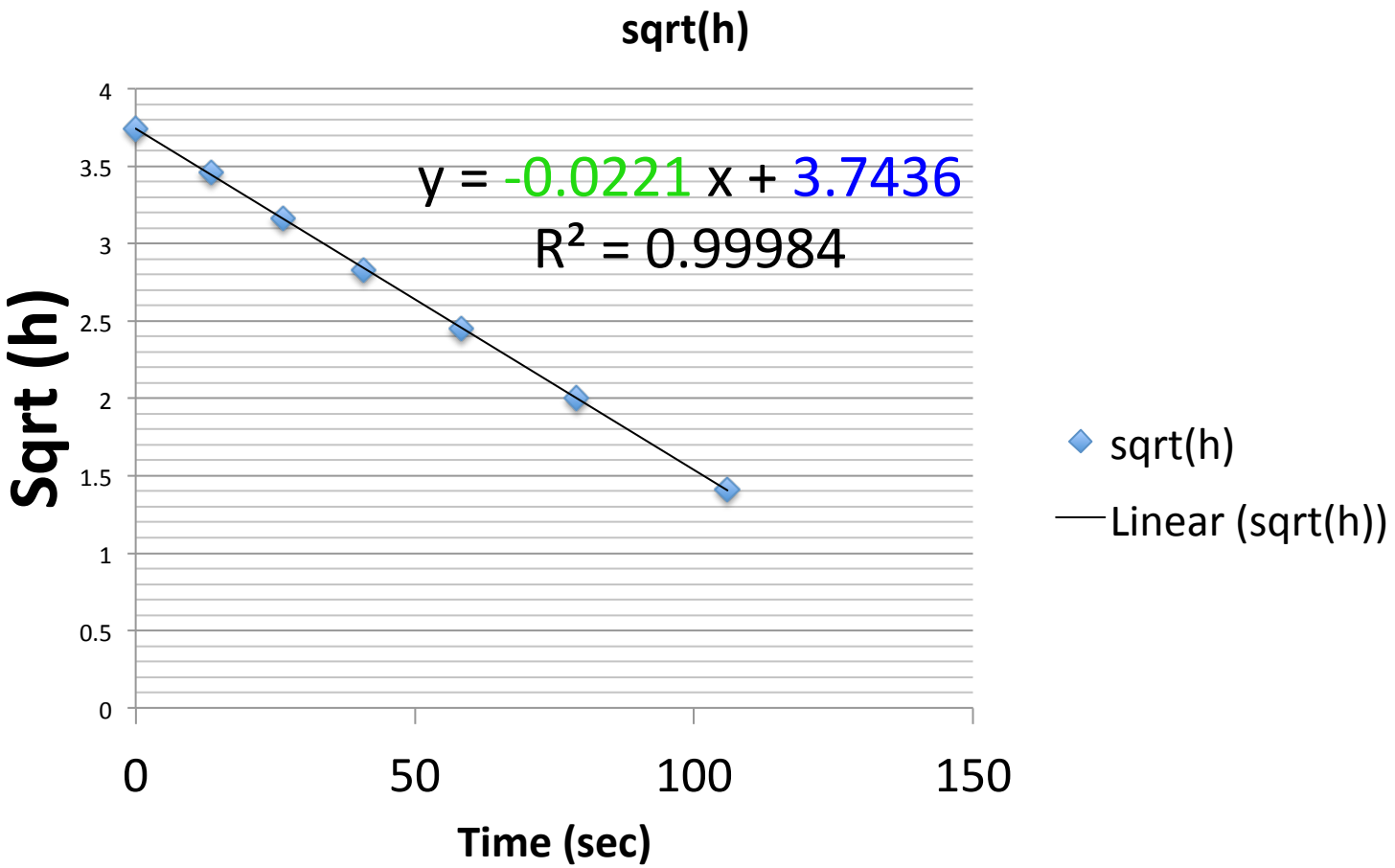
| | A | B | |
|---|---|---|---|
| 1 | t | height(data) | t |
| 2 | 0 | 14 | |
| 3 | 13.46 | 12 | |
| 4 | 26.33 | 10 | |
| 5 | 40.8 | 8 | |
| 6 | 58.35 | 6 | |
| 7 | 79 | 4 | |
| 8 | 106 | 2 | |

transformed data

| D |
|---|
| sqrt(h) |
| 3.74165739 |
| 3.46410162 |
| 3.16227766 |
| 2.82842712 |
| 2.44948974 |
| 2 |
| 1.41421356 |

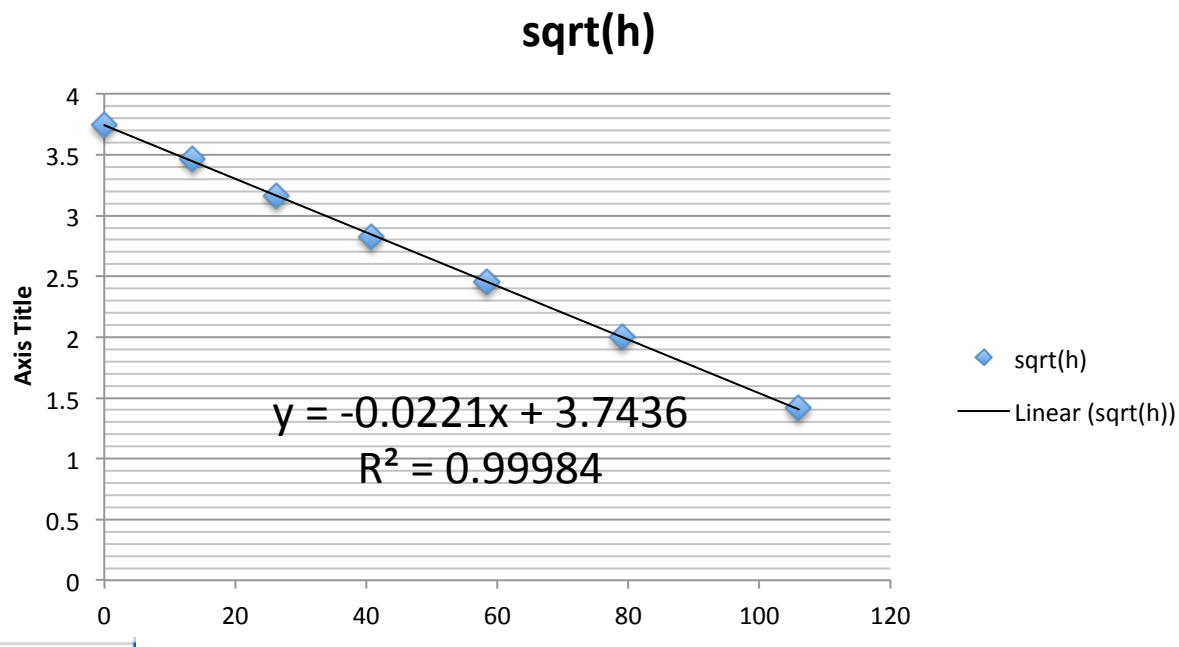
# Linear plot of transformed variables

- Slope = -0.0221 intercept = 3.7436



# Linear plot

- Sqrt(h) vs t



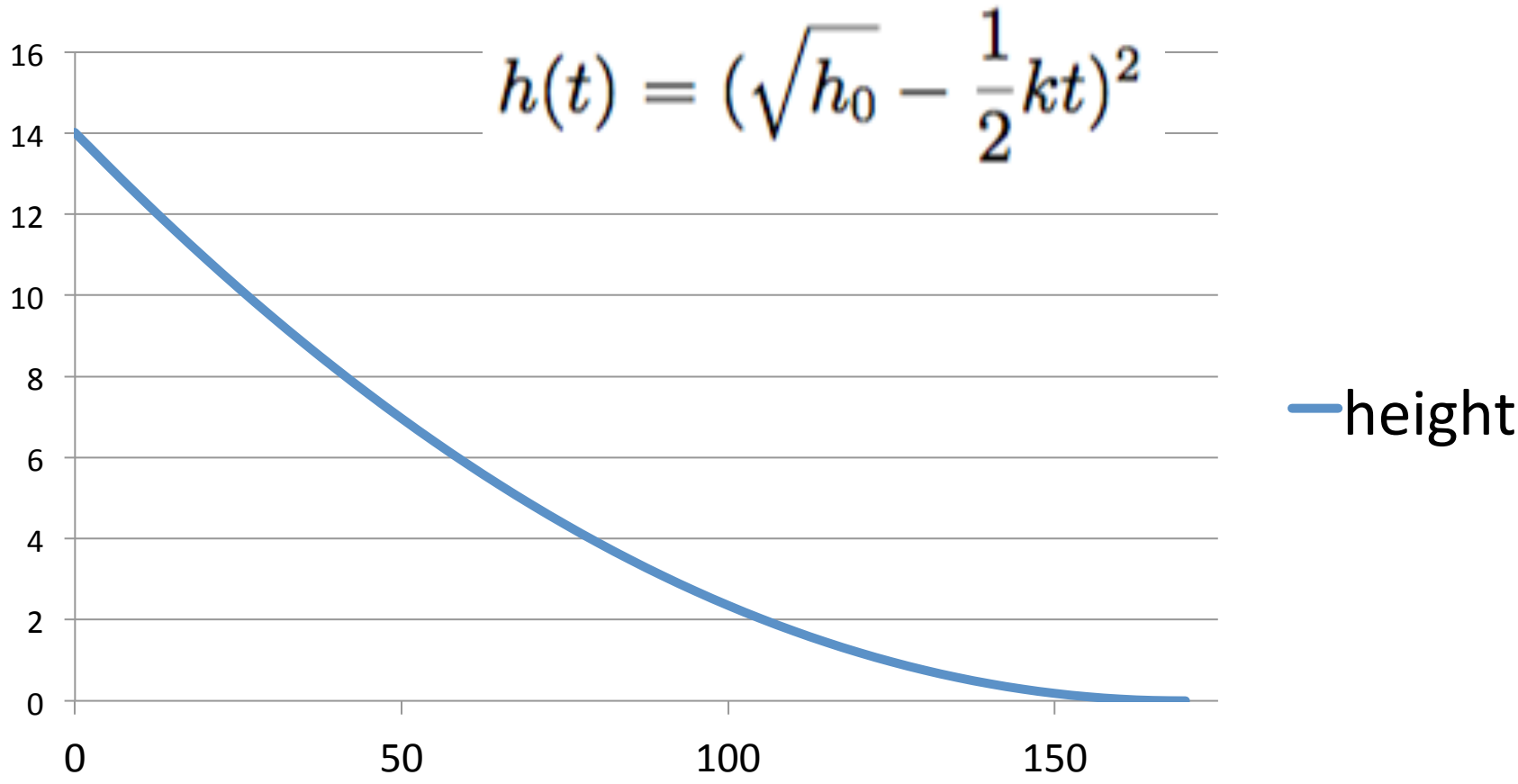

| | | |
|---|---|---|
| 12 | FitParameters | |
| 13 | h0 | 14.014541 |
| 14 | k | 0.0442 |
| 15 | | |
| 16 | empty time | |
| 17 | 169.393665 | |

=3.7436^2

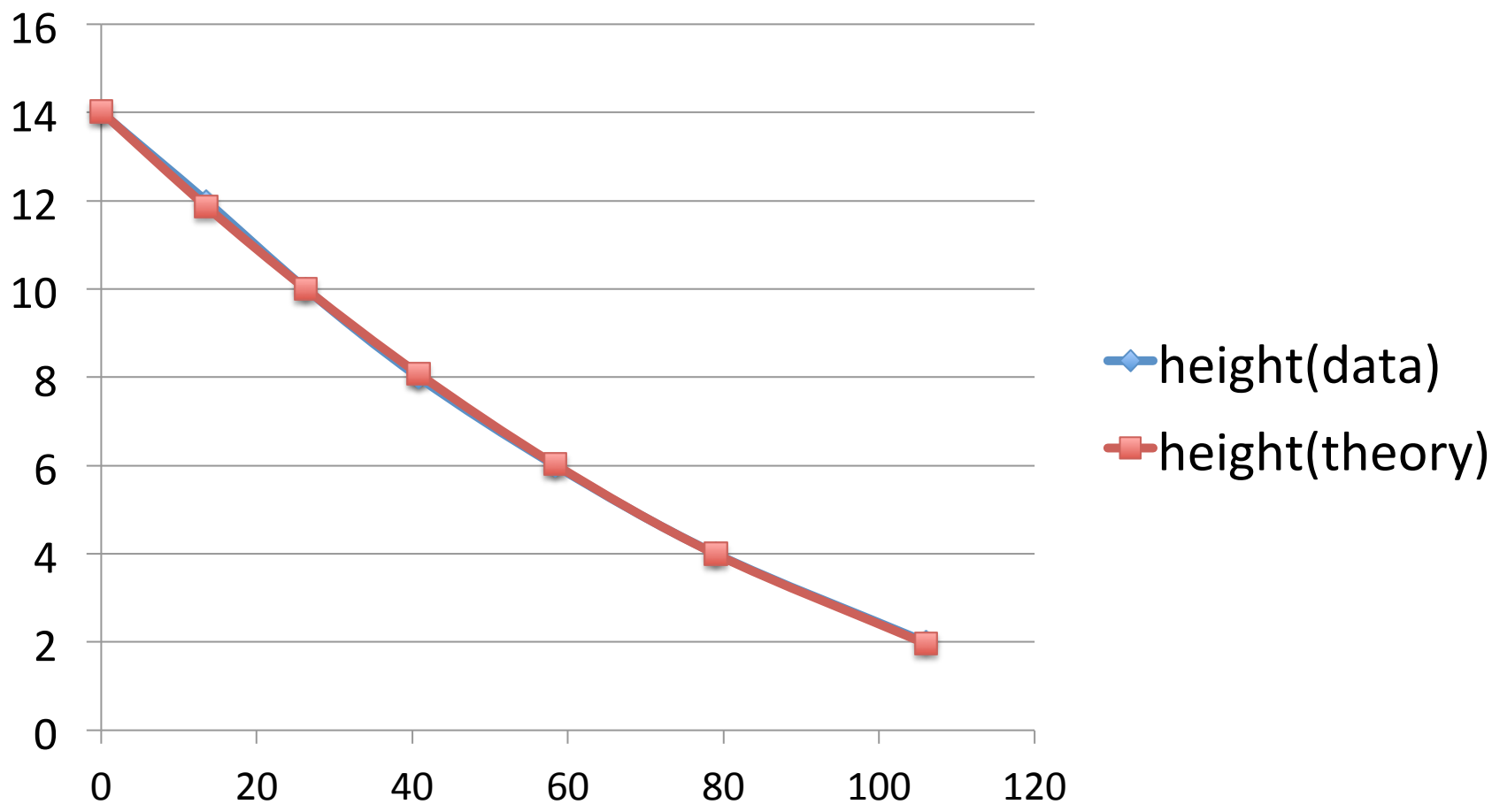
=2*0.0221

=2*SQRT(B13)/B14

# The full function h(t) vs t



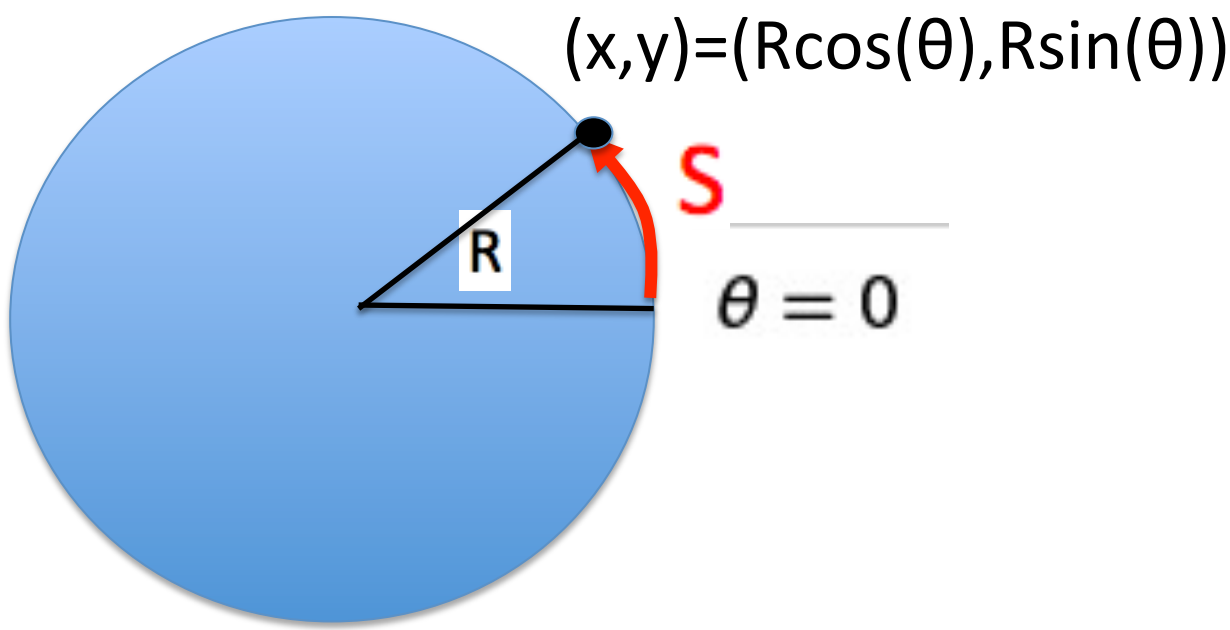
# Data fits the prediction amazingly well



# Trigonometric functions

And revolutions around a circle

# Point on circle of radius R

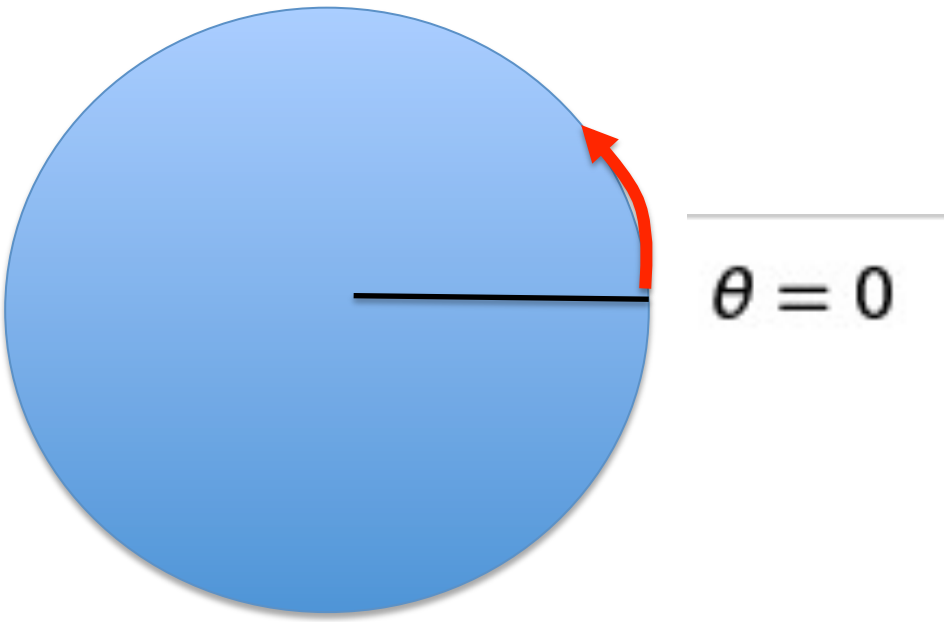



- One revolution = 1 cycle= 2 π radians
- The arc length S corresponding to angle θ is

$$S = R\,\theta$$

# Convention

- Angles increase counterclockwise

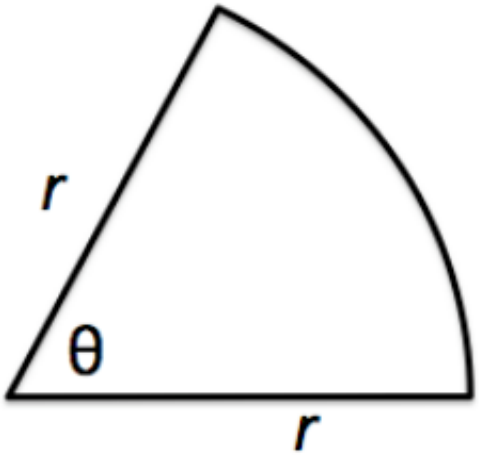


2010 December Q7

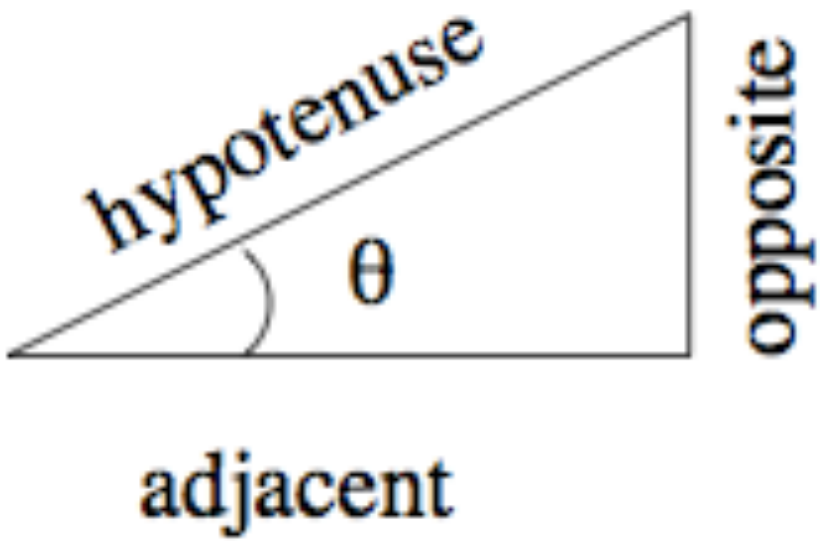
# Review Problem 1

A sector of a circle with radius r and opening angle has area $A = r^2\theta/2$.

Find and for the sector with the smallest perimeter, given that the area . Note that the perimeter consists of two radii and a circular arc.



# Trigonometry and right triangles

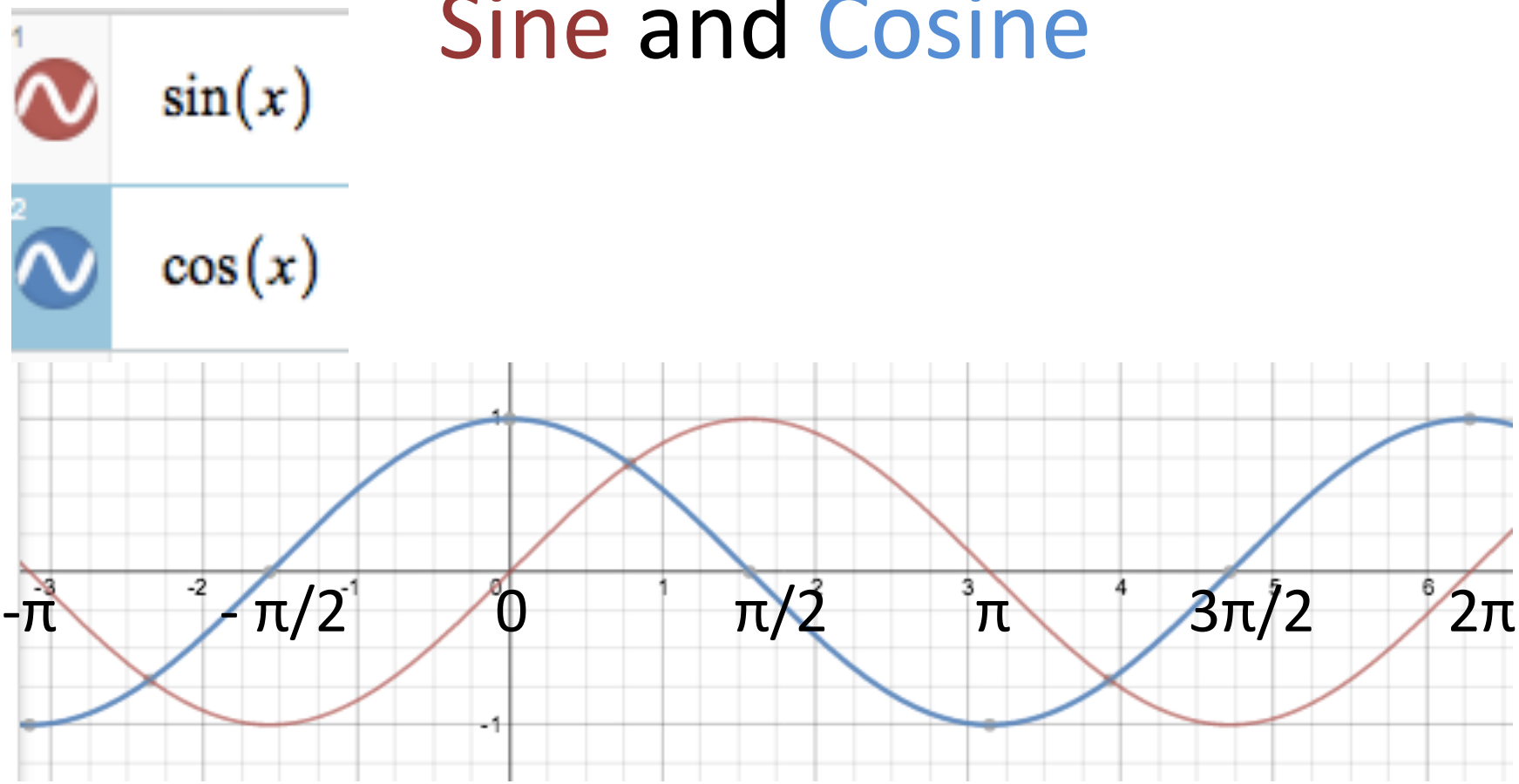



$\sin\theta = \text{opp/hyp}$

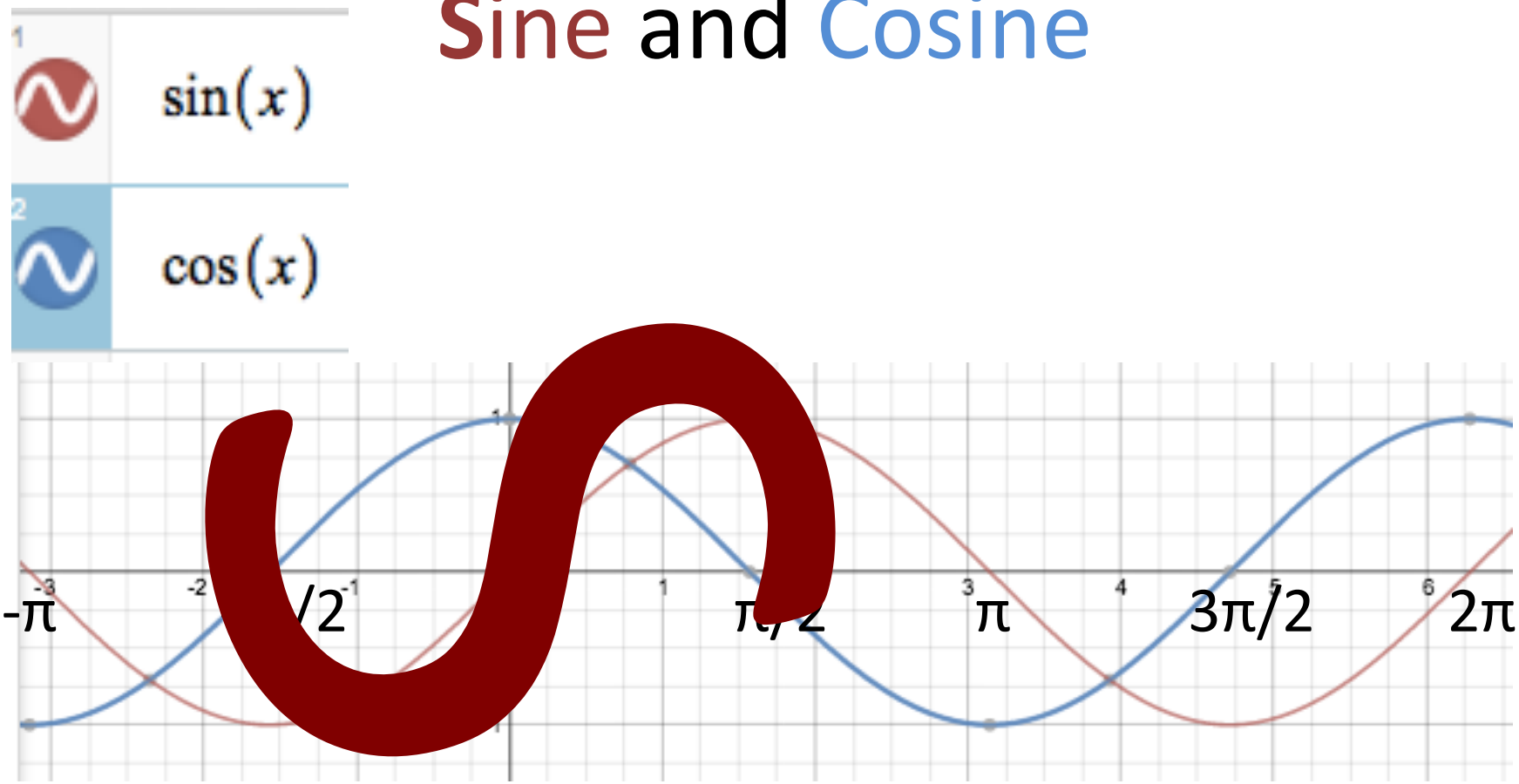
$\cos\theta = \text{adj/hyp}$

$\tan\theta = \text{opp/adj}$

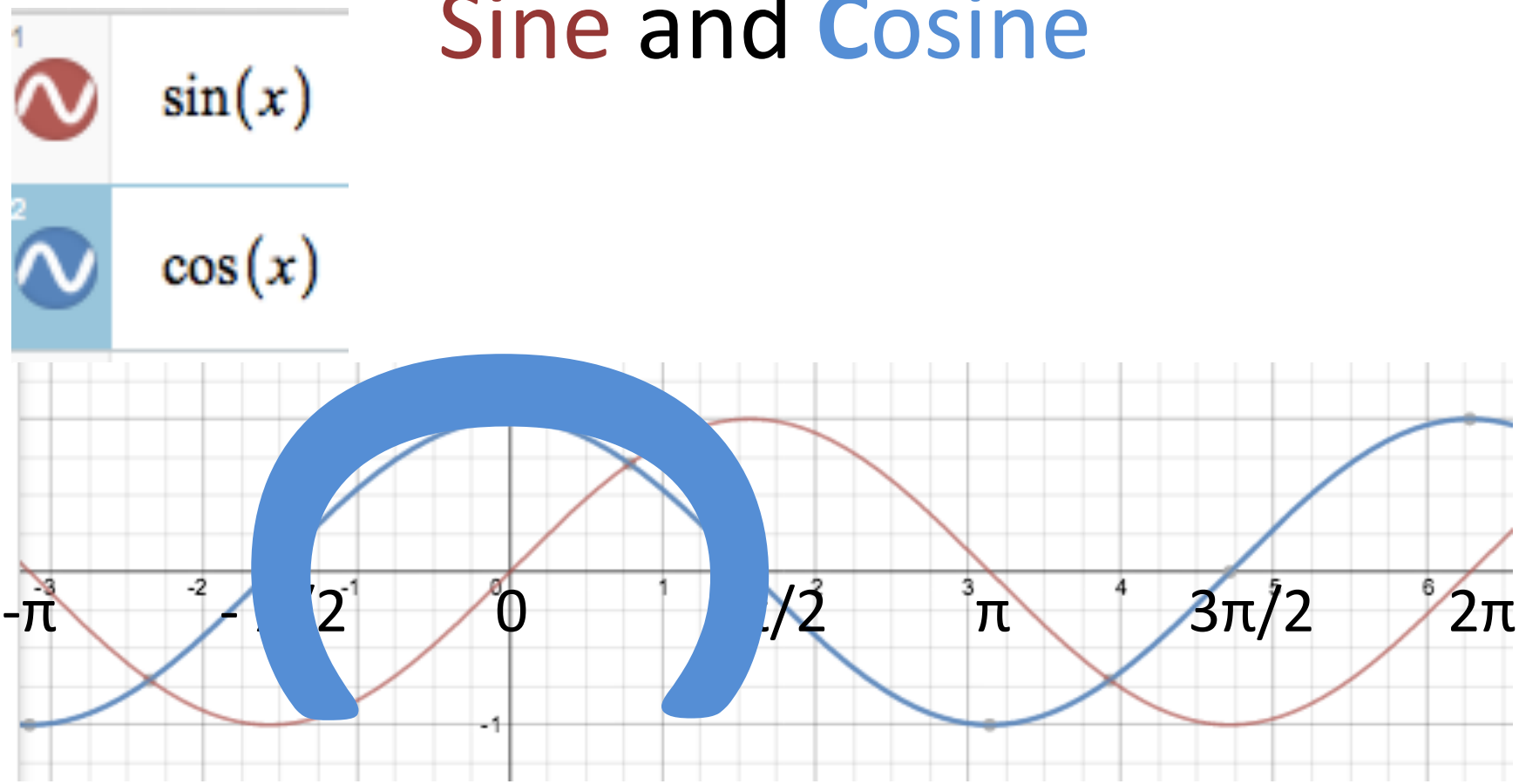
# Sine and Cosine



# Sine and Cosine



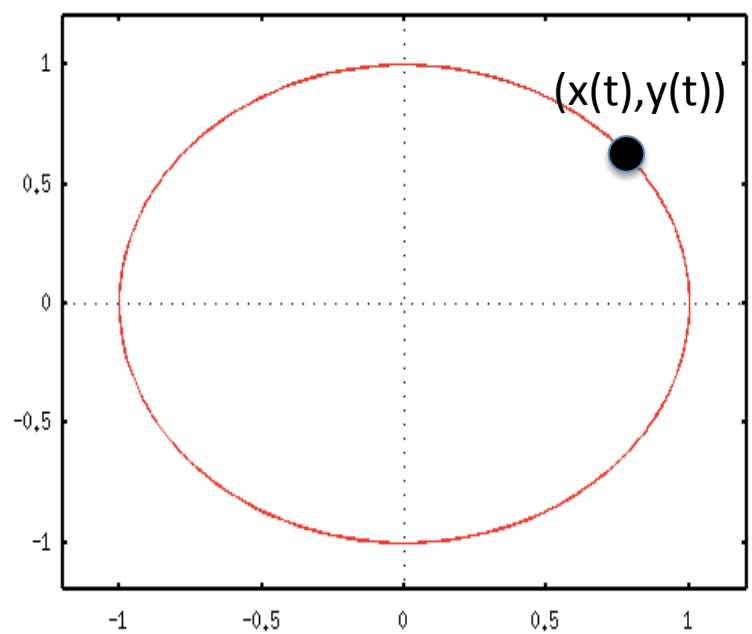
# Sine and Cosine



# Trigonometric functions sin(t), cos(t)

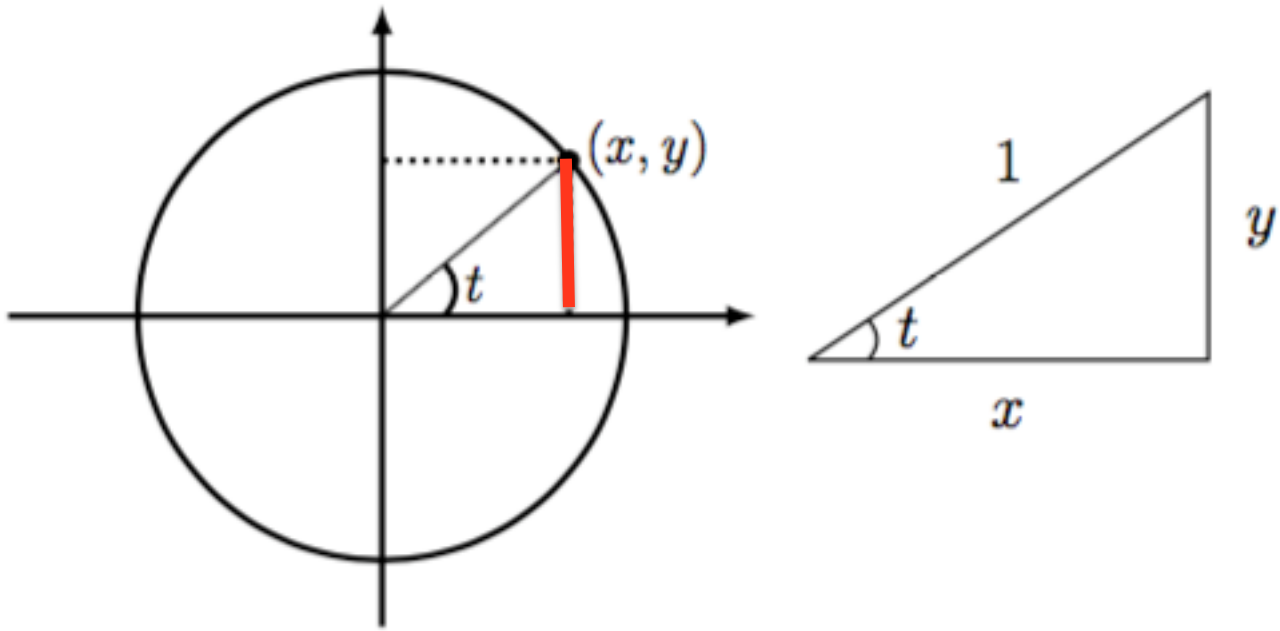
What's special about these functions?

- Classic "periodic functions"
- Describe motion around circle
- Specially "nice" derivs!
- Close relatives..



# Point moving around a circle

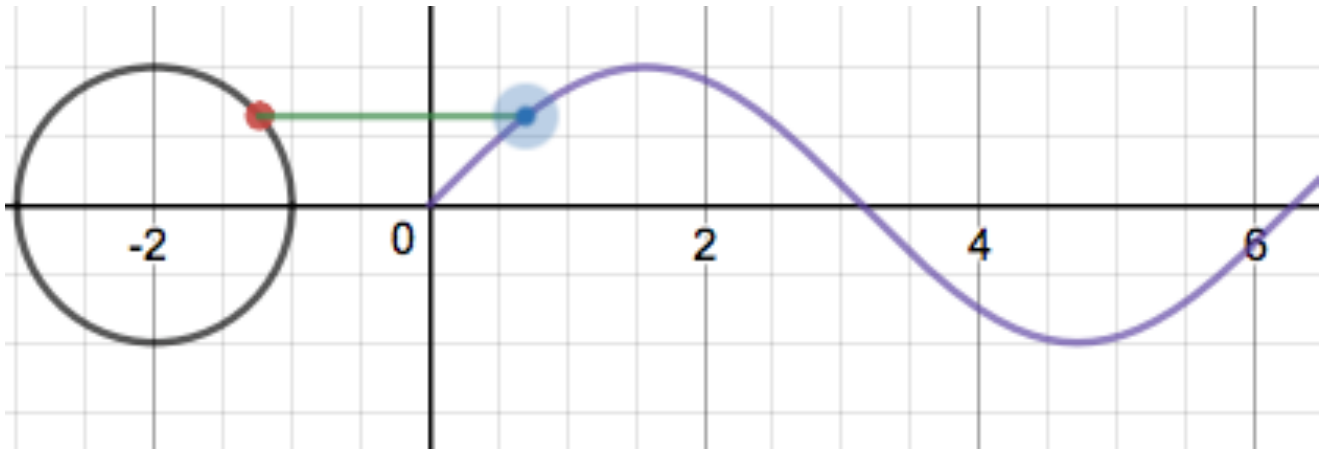
- Sin(t) is the y coordinate of a point moving around a unit circle




$$\sin(t) = \frac{y}{1} = y, \quad \cos(t) = \frac{x}{1} = x$$

# Desmos

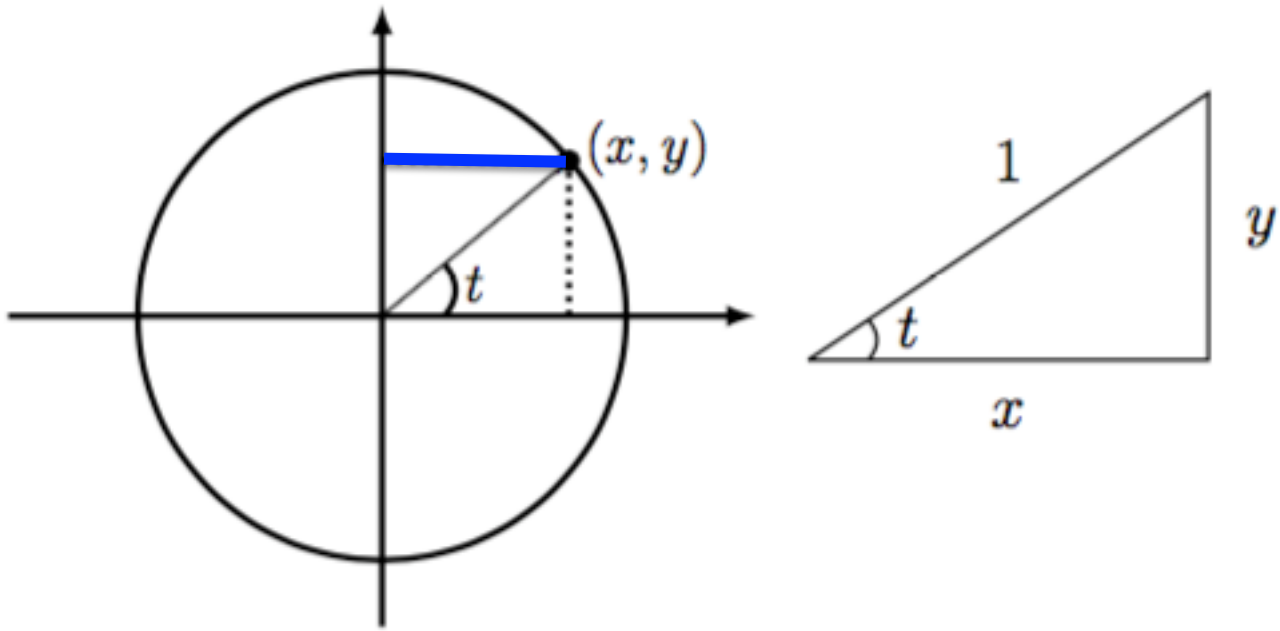
- See link on p 306 of OpenBook.




A demonstration of the link between motion on a circle and the function $y = \sin(x)$. Click on the arrow left of the parameter $a$ or shift the slider on $a$ to see the moving point.

# Point moving around a circle

- Cos(t) is the x coordinate of a point moving around a unit circle




$$\sin(t) = \frac{y}{1} = y, \quad \cos(t) = \frac{x}{1} = x$$

# Trig identity:

Eqn of circle of radius 1: $x^2 + y^2 = 1$

Point on that circle: $(\cos(t), \sin(t))$

Implies special relationship:

$$\sin^2(t) + \cos^2(t) = 1$$

# Motion around a circle of radius R

Angle (theta) increases $$\frac{d\theta}{dt} = \omega$$

If the angular speed is constant then

$$\theta(t) = \omega t$$

And

$$(x(t), y(t)) = (R\cos(\omega t), R\sin(\omega t))$$

# Periodic functions

A function is said to be **periodic** with period T if

$$f(t) = f(t + T).$$

For Trig functions such as sine and cosine, you should be able to identify the period, frequency, amplitude, and mean.

# Frequency and period

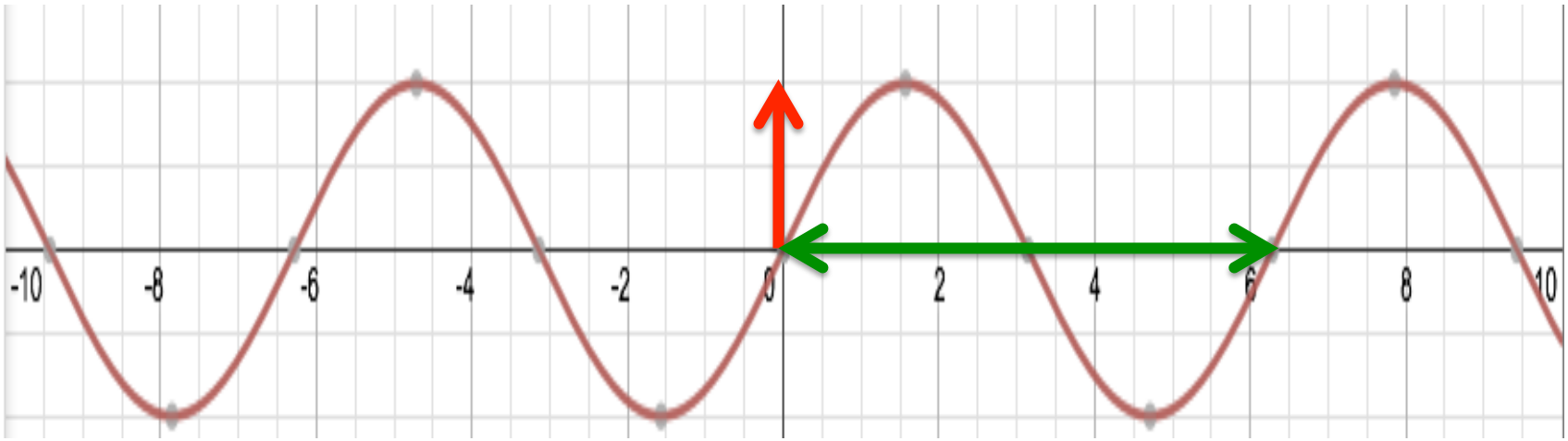
The frequency $\omega$ (radians/time)

One cycle (1 full revolution) = $2\pi$ radians

The period (time to complete 1 cycle:

$$T = 2\pi/\omega$$

# Period (T), amplitude (A)




A= Amplitude = 1

T=Period = 2π

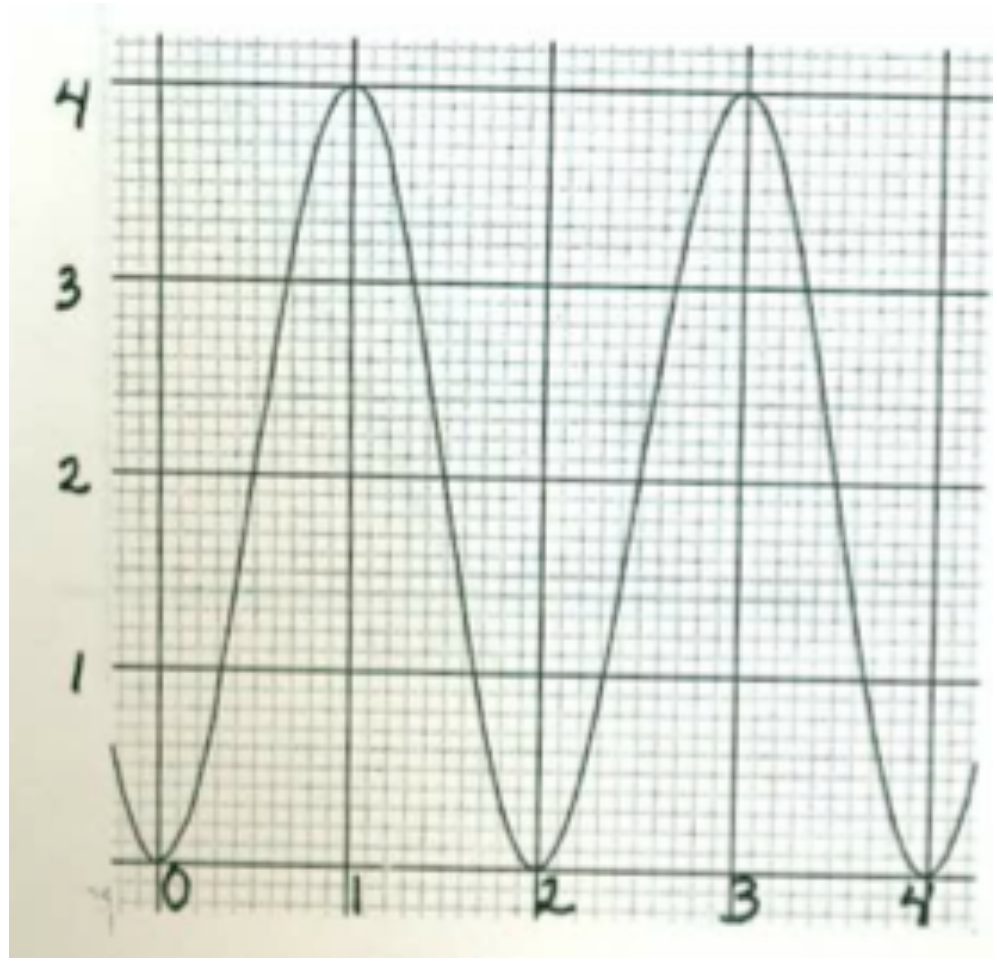
Frequency: $\omega = \frac{2\pi}{T}$

# Problem 4

Shown in the figure is the graph of the function

$$y = R\sin(\omega(t-b)) + c.$$

From the graph, determine the values of the constants $R, \omega, b, c.$



# Multiple Choice Q

**1.8**: Which of the following functions corresponds to the one plotted to the right?

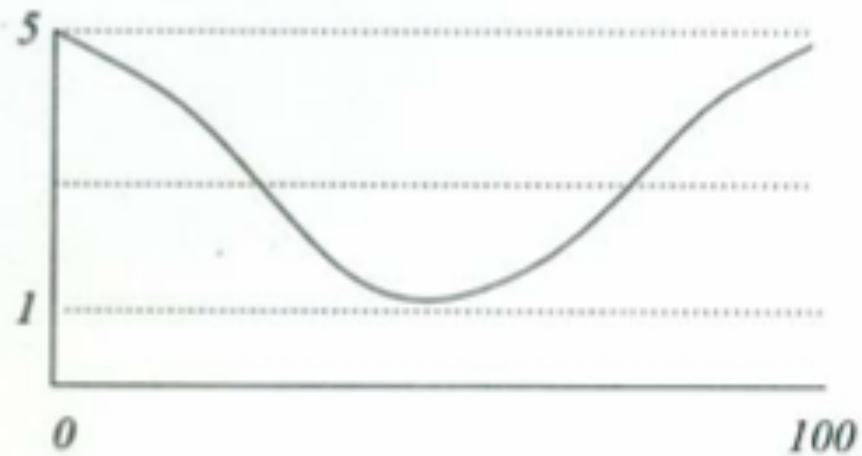
(a) $f(x) = 3 + 2\sin\left(\frac{\pi}{50}(t+25)\right)$

(b) $f(x) = 5\sin\left(\frac{\pi}{100}(t+25)\right) - 1$

(c) $f(x) = 3 + 2\sin\left(\frac{\pi}{50}(t-25)\right)$

(d) $f(x) = 3 + \sin\left(\frac{\pi}{100}(t-25)\right)$

(e) $f(x) = 1 + 4\sin\left(\frac{\pi}{100}(t-25)\right)$



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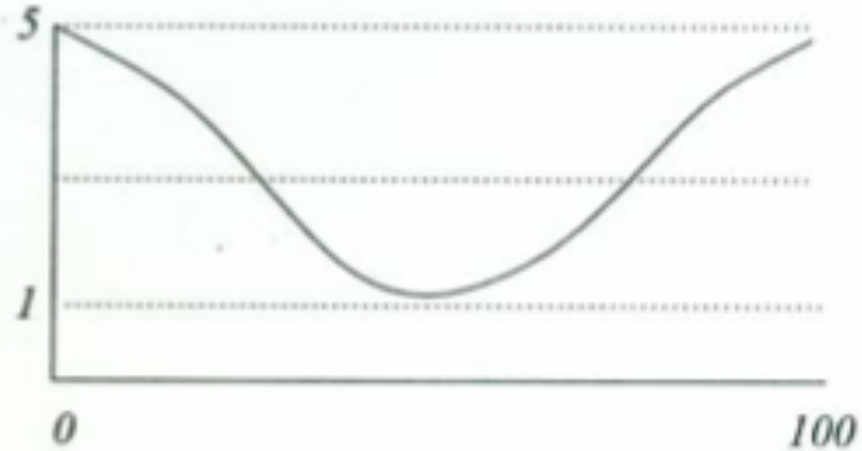
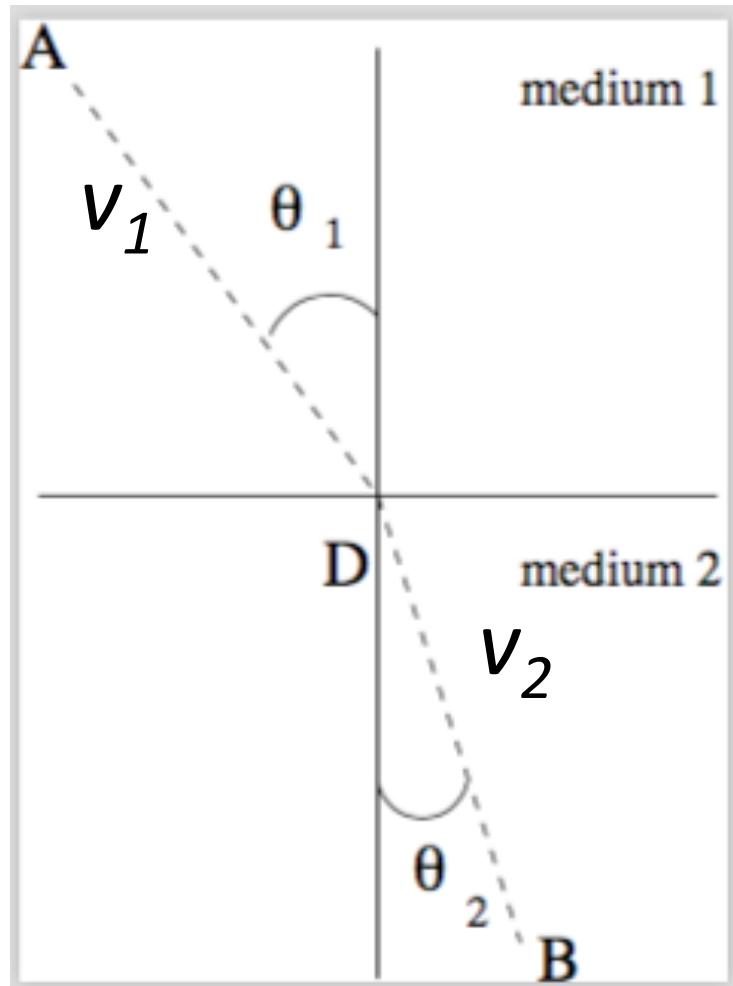



This answer was corrected after class was over.

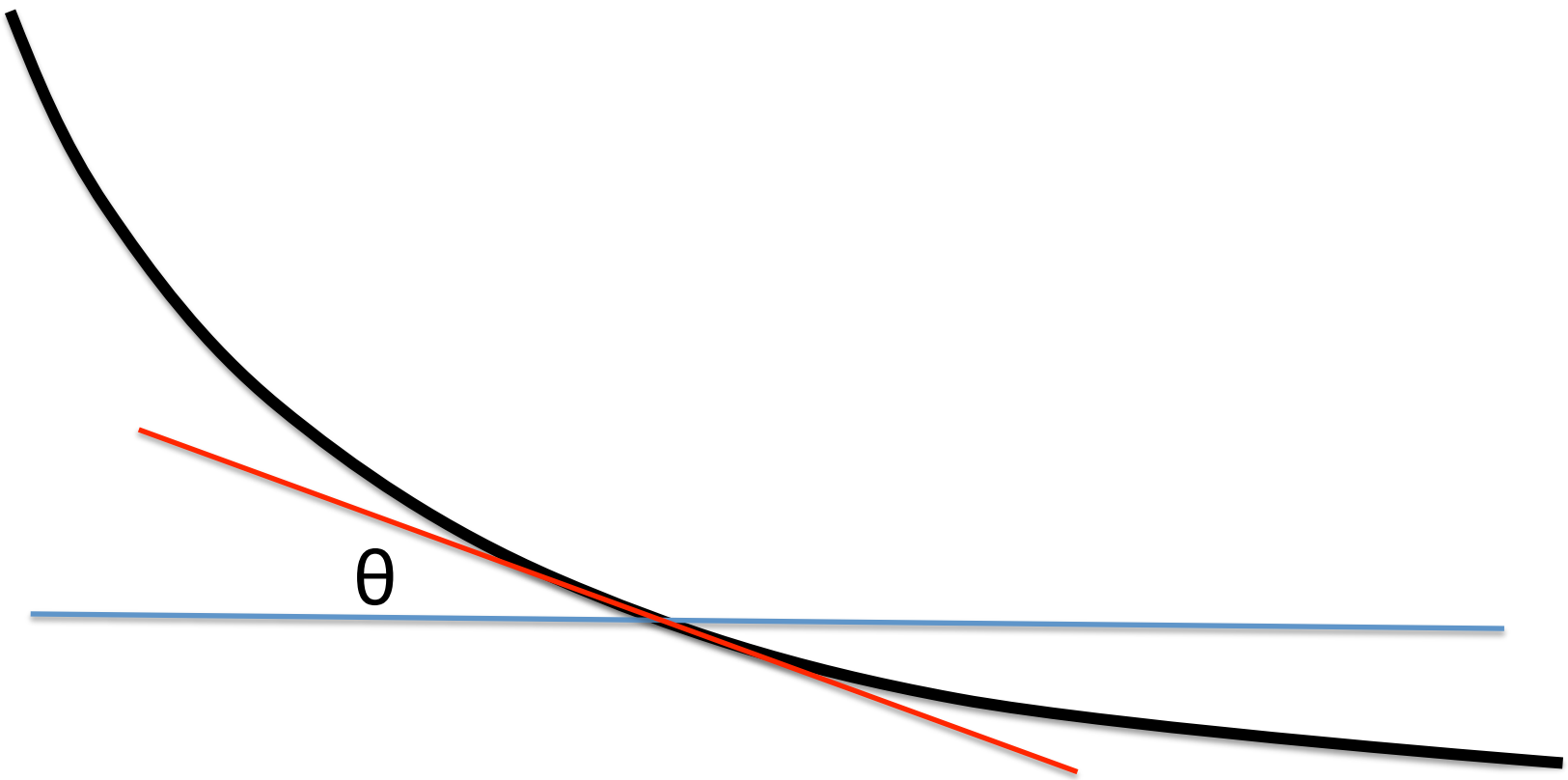
# Optimization Problem (review+trig)

Show that Snell's Law solves the problem of finding the least time to travel between two points with given velocities.

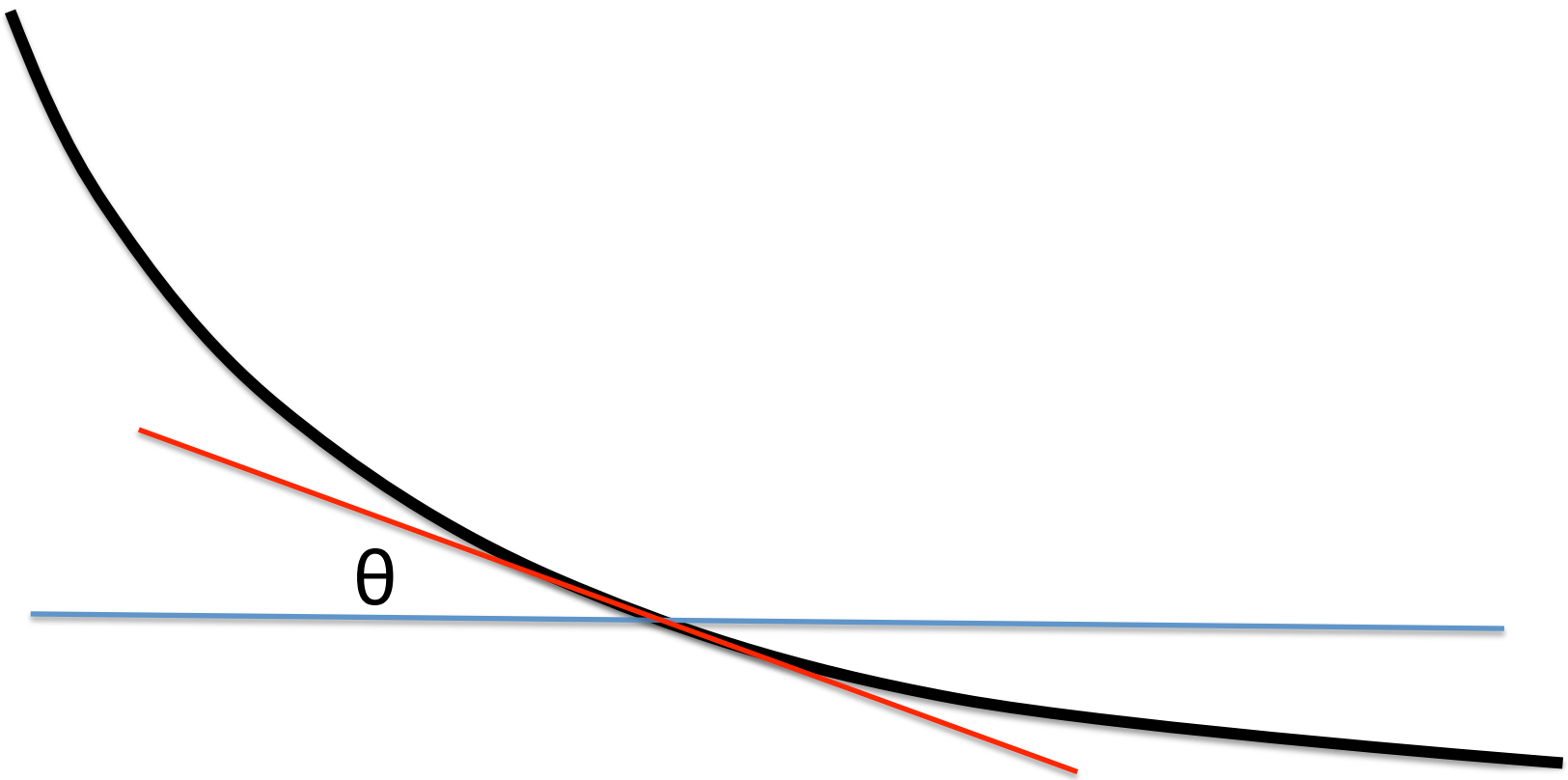
$$\frac{1}{v_1}\sin(\theta_1) = \frac{1}{v_2}\sin(\theta_2)$$



# Connection of tan to tangent line?



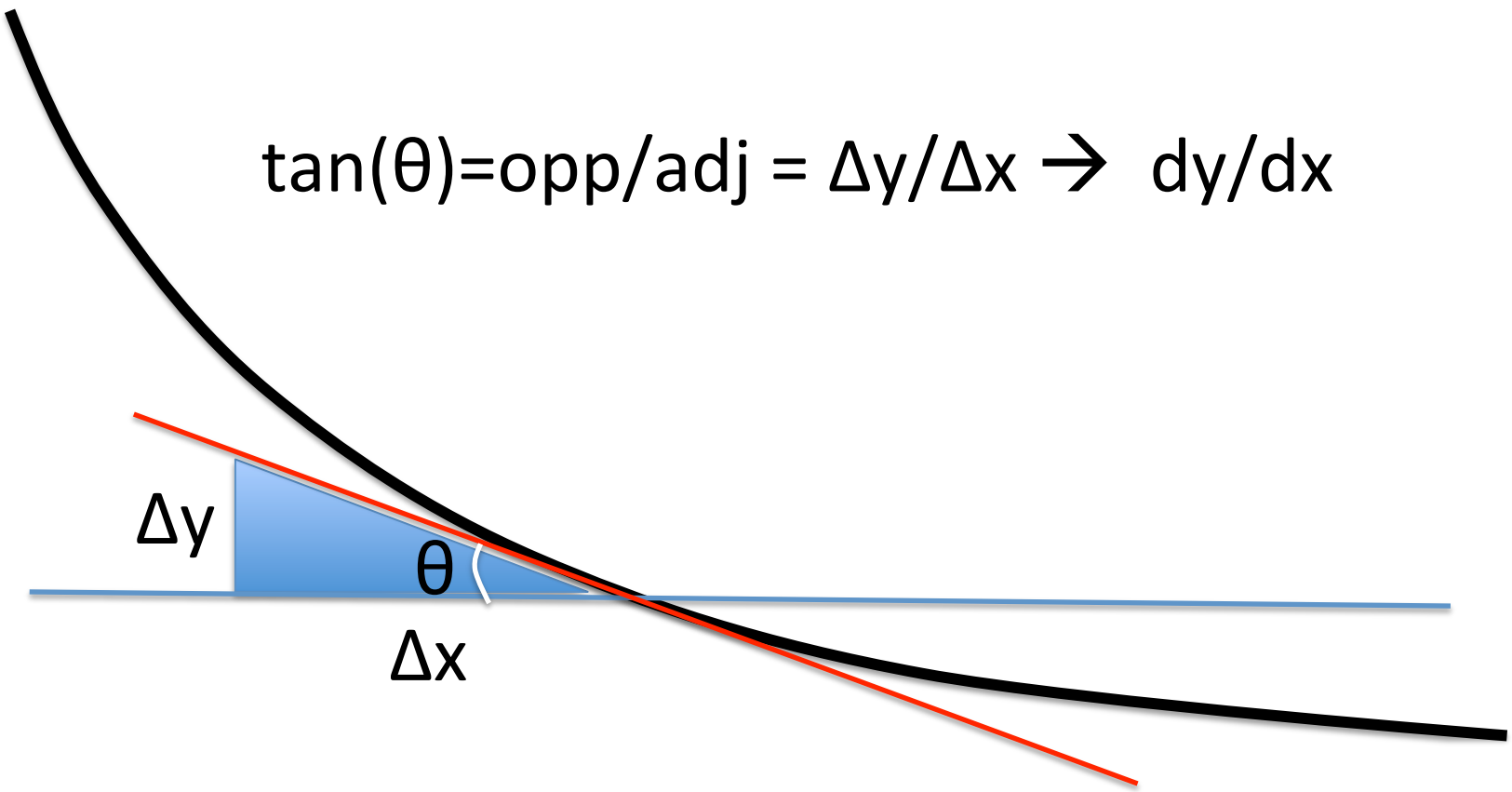
# Slope of tangent line=tan(θ)



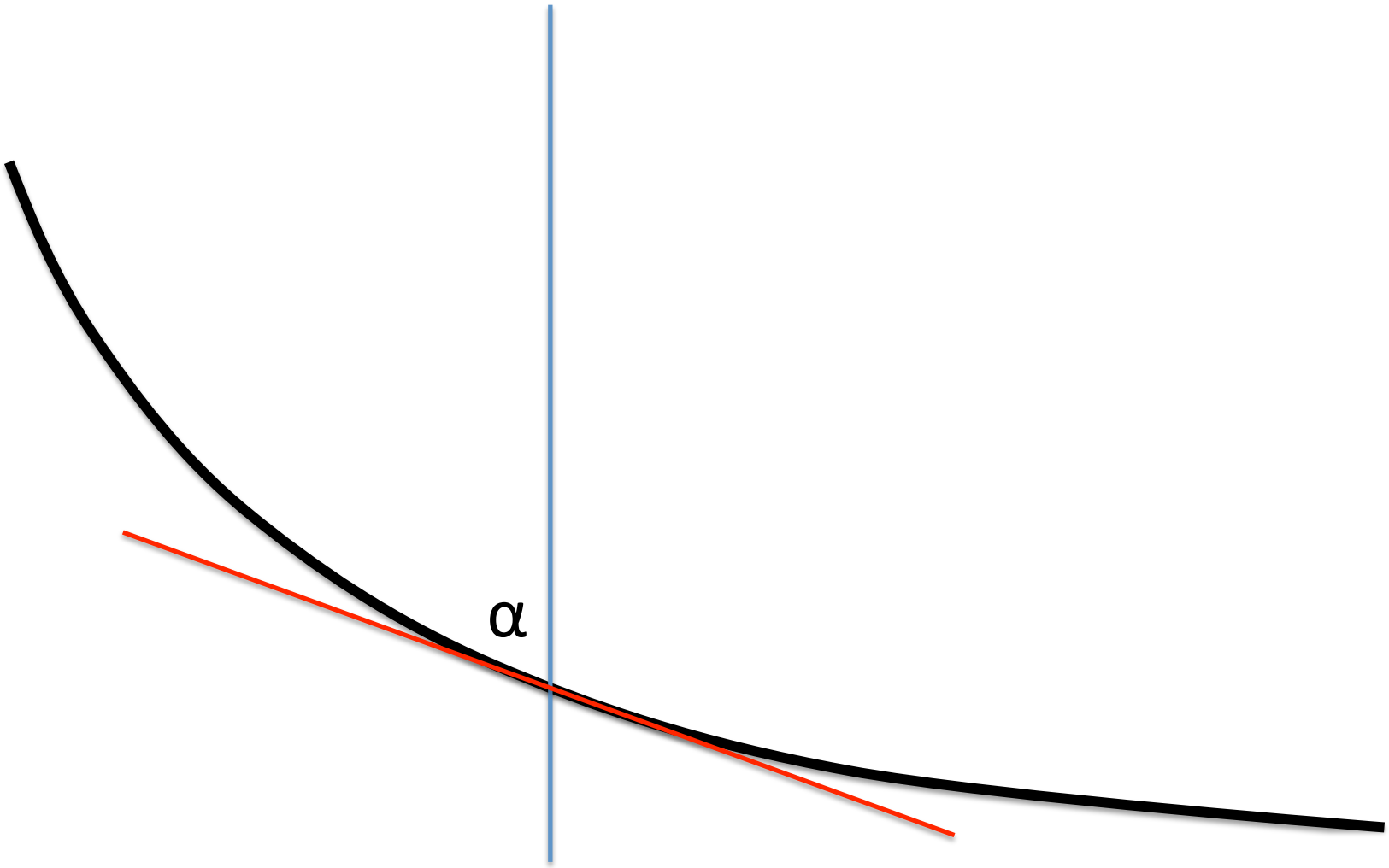
# Slope of tangent line=tan(θ)

Where θ = angle between curve and horizontal

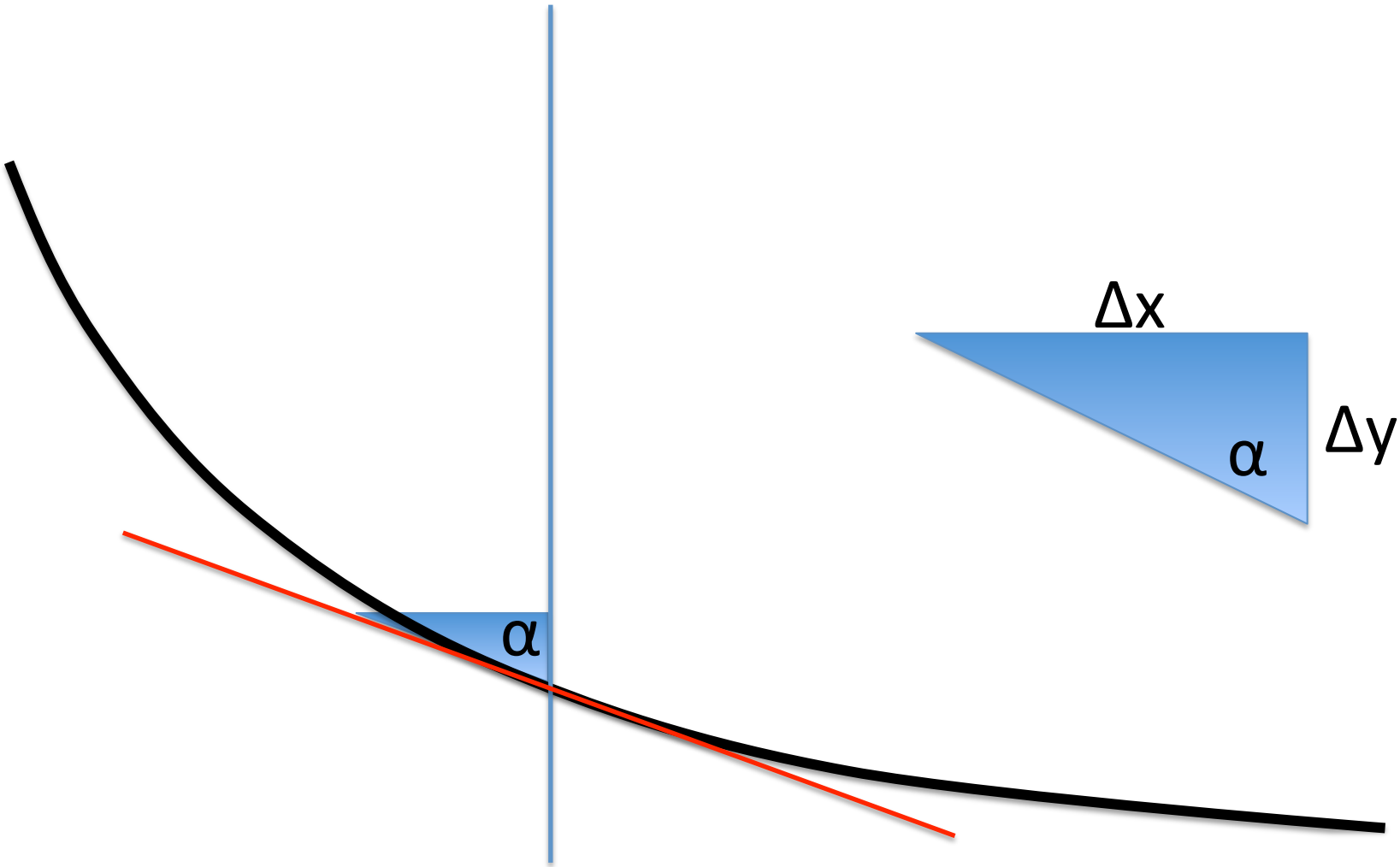
$\tan(\theta)=opp/adj = \Delta y/\Delta x \rightarrow dy/dx$



# Relate sin(α) to dy/dx



# Relate sin(α) to dy/dx



# Derivatives of cosine and sine

Cosine:

$$\boxed{\frac{d\cos(t)}{dt} = -\sin(t)}$$

Sine:

$$\frac{d\sin(t)}{dt} = \cos(t)$$

(Can show this using the definition of the derivative)

# Derivative of sin(x)

See course Notes Section 15.1.1

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
\frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right)+\cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\
&= \cos(x).
\end{aligned}
$$

# Uses Trig identity

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
\frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \lim_{h\to 0} \left( \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \right) \\
&= \sin(x)\left( \lim_{h\to 0} \frac{\cos(h)-1}{h} \right) + \cos(x)\left( \lim_{h\to 0} \frac{\sin(h)}{h} \right) \\
&= \cos(x).
\end{aligned}$$

# Uses two limits:

See course Notes Section 15.1.

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
\frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right) + \cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\
&= \cos(x).
\end{aligned}$$

# Uses two limits:

See cours 1.1

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

$$f'(x) =$$

$$\frac{d\sin(x)}{dx} =$$

$$= \lim_{h\to 0} \frac{\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left( \sin \frac{-1}{h} + \cos(x)\frac{\sin(h)}{h} \right)$$

$$= \sin(x) \left( \lim_{h\to 0} \frac{\cos(h)-1}{h} \right) + \cos(x) \left( \quad 1 \quad \right)$$

$$= \cos(x).$$

# Uses two limits:

See course Notes Section 15.1.1

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
\frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\left(0\right) + \cos(x)\left(1\right) \\
&= \cos(x).
\end{aligned}
$$

# Derivatives of sine and cosine

Cosine: $$\frac{d\cos(t)}{dt} = -\sin(t)$$

sine: $$\frac{d\sin(t)}{dt} = \cos(t)$$

$$\boxed{\begin{aligned} dx/dt &= -\ y \\ dy/dt &= \ \ x \end{aligned}}$$

(Special relationship between these functions means that they satisfy the Juliet-Romeo equations!)

# Problem 2:

Consider a point on the rim of a circle of radius R=3. If the point revolves around the circle at the rate of 0.5 revolutions per minute, at what rate is its x coordinate changing?

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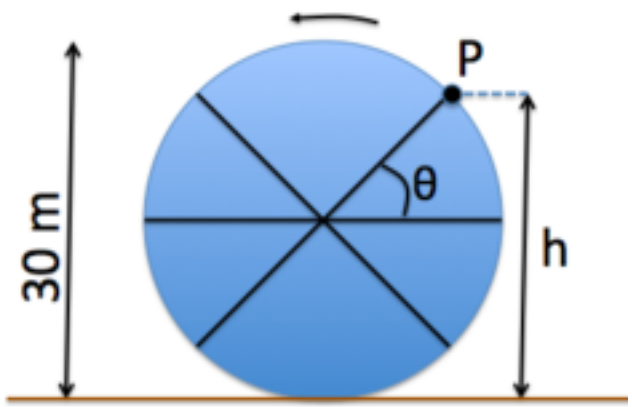
# Problem 3

A ferris wheel 30 meters high turns counterclockwise at a uniform rate of 3 revolutions per minute. Find the rate of change in the altitude (h) of a passenger at the instant when they are located at point P. Use the fact that $\theta = \pi/3$ radians at that instant.



# Other practice questions:

- See our Question Challenge wiki

http://wiki.ubc.ca/Course:MATH102/Question_Challenge