Applications of simple differential equations to biology

Qualitative methods applied to a predator-prey system
Aphid and ladybugs

Let \( x = \) population size of aphids.
Let \( y = \) population of ladybugs
\( G(x) = \) Aphids population growth rate
\( P(x,y) = \) Aphid predation rate by ladybugs.
Aphid and ladybugs

Let $x =$ population size of aphids.
Let $y =$ population of ladybugs

$G(x) =$Aphids population growth rate $= r \times x$

$P(x,y) =$Aphid predation rate by ladybugs

\[
P(x, y) = y \cdot \left( \frac{30x^2}{20^2 + x^2} \right)
\]
Previously: Aphid and ladybugs

When does growth rate balance predation rate?

\[ G(x) = P(x) \]
Now:

What happens to the aphid population when these do NOT match!

The aphid population will change!

We can use a differential equations to track that change
Rate of change of aphid population

dx/dt = Growth rate – predation rate

= \ G(x) - P(x,y)

Net growth rate

F(x,y)

We can determine how the aphid population changes by drawing a slope field or state-space diagram of the equation \( dx/dt = F(x) \)
Example: 1 ladybug (y=1)

Predation rate = \( P(x,y) = K \frac{x^2}{a^2 + x^2} \)

Growth rate = \( G(x) = rx \)

Net growth rate \( F(x) = G(x) - P(x) \)

\[ = rx - K \frac{x^2}{a^2 + x^2} \]
Problem is now:

Sketch the qualitative behaviour for the differential equation

\[
\frac{dx}{dt} = rx - K \frac{x^2}{a^2 + x^2} = F(x)
\]
Predation and growth rate:

\[ P(x) = K \frac{x^2}{a^2 + x^2} \]

\[ G(x) = rx \]
Steady states:

Aphid population does not change if $G(x) = P(x)$
Predictions:

Aphid population increases if $G(x) > P(x)$
Aphid population decreases if $G(x) < P(x)$
Directions of flow:

The sign of \( G(x) - P(x) \) indicates if \( x \) increases or decreases.
Stability of steady states

Unstable  Stable  Unstable

Aphid population, $x$
Alternatively:

- Sketch the function $F(x)$

\[
\frac{dx}{dt} = rx - K \frac{x^2}{a^2 + x^2}
\]

$= F(x)$
\[ F(x) = G(x) - P(x) \]

- Both functions:
  \[ P(x) = K \frac{x^2}{a^2 + x^2} \]
  \[ G(x) = rx \]

- Subtracting:
  \[ F(x) = G(x) - P(x) \]
Net growth rate

Aphid population, x
State-space diagram:

Aphid population, $x$
What does this tell us?

(A) One ladybug can control the aphid population
(B) The aphid population will always grow
(C) The ladybug population has a carrying capacity
(D) If there are too many aphids initially, then one ladybug will not be able to control their level
(E) A large aphid population cannot be sustained due to resource limitations.
What does this tell us?

(A) One ladybug can control the aphid population
(B) The aphid population will always grow
(C) The ladybug population has a carrying capacity
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(E) A large aphid population cannot be sustained due to resource limitations.
For further study:

(1) Draw a slope field and some solution curves for the same system

(2) How would your conclusions change for a type II predator response?
What if.. 

The growth rate of aphids is really high (1) or really low (2)?

(1)

(2)
And now..

• Here is some cool stuff

That won’t be on the final exam!! (just for fun)
What if the predator population also changes?

Aphids: \[ \frac{dx}{dt} = \text{rate of (birth} - \text{mortality)} -\text{rate of predation} \]

Ladybugs: \[ \frac{dy}{dt} = \text{rate of birth} - \text{rate of mortality} \]
What if the predator population also changes?

Aphids: \( \frac{dx}{dt} = \text{rate of (birth} - \text{mortality)} - \text{rate of predation} \)

Ladybugs: \( \frac{dy}{dt} = \text{rate of birth} - \text{rate of mortality} \)

The more I eat, the more babies I make
What if the predator population also changes?

Aphids:  \[ \frac{dx}{dt} = r x - P(x, y) \]

Ladybugs:  \[ \frac{dy}{dt} = k P(x, y) - d y \]

The populations are linked to one another!
“Coupled system of differential equations”
Keep track of BOTH populations

Aphids: \[ \frac{dx}{dt} = r x - P(x, y) \]

Ladybugs: \[ \frac{dy}{dt} = k P(x, y) - d y \]

We can still get an idea of what happens using the concept of slope field.
Constructing a slope field

Pick any point:

\[
\frac{dx}{dt} = r x - P(x, y) \\
\frac{dy}{dt} = k P(x, y) - dy
\]
Keep track of both $x$ and $y$

Pick a point with known $x$, $y$ coord:

$$\frac{dx}{dt} = r x - P(x,y)$$
$$\frac{dy}{dt} = k P(x, y) - dy$$
Determine values of $\frac{dx}{dt}$, $\frac{dy}{dt}$

\[ \frac{dx}{dt} = r \cdot x - P(x, y) \]
\[ \frac{dy}{dt} = k \cdot P(x, y) - dy \]
Draw an arrow representing changing $x$ and $y$ at this point

$\frac{dx}{dt} = r x - P(x,y)$

$\frac{dy}{dt} = k P(x, y) - dy$

See Appendix for an example of this
Continue this at many points:

\[ \frac{dx}{dt} = r x - P(x,y) \]
\[ \frac{dy}{dt} = k P(x, y) - dy \]
Direction field

\[ \frac{dx}{dt} = r \cdot x - P(x) \cdot y \]
\[ \frac{dy}{dt} = k \cdot P(x) \cdot y - d \cdot y \]

(dx/dt, dy/dt)
Using the direction field, draw solution curves:

\[
\frac{dx}{dt} = r x - P(x,y) \\
\frac{dy}{dt} = k P(x, y) - dy
\]
Solution curve

dx/dt = r x − P(x) y

dy/dt = k P(x) y − d y

Aphids, x

ladybugs y
dx/dt = r x – P(x) y
dy/dt = k P(x) y – d y

Solution curve

Aphids, x

ladybugs y

Now add a time axis
Cycles of aphids and ladybugs!

Aphids $x(t)$

Ladybugs $y(t)$

Time
• Real cycles of predators and prey (lynx and hare)
The solution curves are now in 3D

Curve for \((t, x(t), y(t))\):
Stereoscopic view
Now preparation for quiz 3

• But first, a reminder of a differential equation we saw a while ago.
Flow of fluid out of cylinder

Cylindrical tank of fluid.
Fluid height $h(t)$

DE:

$$\frac{dh}{dt} = -k\sqrt{h},$$

with

$$h(0) = h_0,$$

And

$$k = \frac{a}{A} \sqrt{2g}$$
Flow of fluid out of cylinder

Cylindrical tank of fluid.
Fluid height h(t)

DE: \[
\frac{dh}{dt} = -k\sqrt{h},
\]

Last week we showed how to check that the following is a soln:

\[
h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2
\]
How good is this prediction?

• I was wondering if this theoretical result has any bearing on reality..
• Let us do an experiment..
Data (from experiment)

• In the group-version of the quiz, I’ll ask you to see if this data fits the model. This is meant to be a learning activity (for which you get some bonus points).
Data (from my own home expt)

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<th>height (cm) $h$</th>
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<td>79</td>
<td>4</td>
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<td>104</td>
<td>2</td>
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Data from experiment in class

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Quiz 1

Individual stage

• No notes
• No books
• No computers or calculators

• OK to use scrap paper
Consider a cubic crystal of side length \( x \) and volume \( V = x^3 \). If the volume is increasing at the rate of 3 mm\(^3\) per month, and the crystal remains a cube, at what rate is its side length changing (in mm/month) when its volume is 8 mm\(^3\)?

(A) \( 3^{1/3} \), \quad (B) \( \frac{3^{1/3}}{8} \), \quad (C) \( \frac{9}{64} \), \quad (D) \( \frac{1}{64} \), \quad (E) \( \frac{1}{4} \).
The human population on Earth doubles roughly once every 50 years. If there are 7.5 billion people on Earth now, in how many years will there be 30 billion?

(A) $7.5e^{50t}$ years (B) $7.5e^{t/50}$ years (C) $\frac{7.5}{30}e^{t \ln(2)/50}$ years

(D) 75 years (E) 100 years
Quiz 1

• Q3

Consider the function $y = ax^b$. When I graph $\ln(y)$ versus $\ln(x)$, I see a line with slope $3/4$ and vertical intercept 1. This tells me that

(A) $a = 1, b = 3/4$  (B) $a = 3/4, b = 1$  (C) $a = e, b = 3/4$,

(D) $a = 3/4, b = e$  (E) $a = 1, b = e^{3/4}$. 
Quiz 1

• Q4

Which function is a solution to the differential equation $\frac{dh}{dt} = -k\sqrt{h}$ with $h(0) = h_0$?

(A) $h(t) = \left(\sqrt{h_0} - \frac{1}{2}kt\right)^2$

(B) $h(t) = \left(\sqrt{h_0} - kt^2\right)$

(C) $h(t) = h_0e^{-kt}$

(D) $h(t) = (1 - e^{-kt})$

(E) None of the above
Quiz 1

• Q5

Which of the following differential equations has unstable steady states at $y = \pm 1$ and a stable steady state at $y = 0$? (Hint: consider sketching the state-space diagrams).

(A) $\frac{dy}{dt} = -2(y^3 - y)$

(B) $\frac{dy}{dt} = y^3 - y$

(C) $\frac{dy}{dt} = y(1 - y)^2$

(D) $\frac{dy}{dt} = y^4 - y^2$

(E) None of the above
Quiz 1

• Please hand in Quiz 1
Quiz 1

- Group version:

- Please do use laptops and spreadsheets, notes, books, etc.
**XPP code**

(for the aphid-ladybug computations):

```plaintext
#x=aphids
#y=ladybugs
dx/dt=r*x-y*b*x^2/(k^2+x^2)
dy/dt=y*c*x^2/(k^2+x^2)-d*y

init x=50,y=1

param r=0.5,k=20,b=30,c=5,d=2
@ xplot=t,yplot=y,zplot=x,axes=3d
@ xmin=0,xmax=100,ymin=0,ymax=5,zmin=0,zmax=60
@ xlo=-2,ylo=-2,xhi=2,yhi=2
@ phi=60
done
```

XPP is free software written by my friend Bard Ermentrout. See me for more information.