Applications of simple differential equations to biology

Qualitative methods applied to a predator-prey system

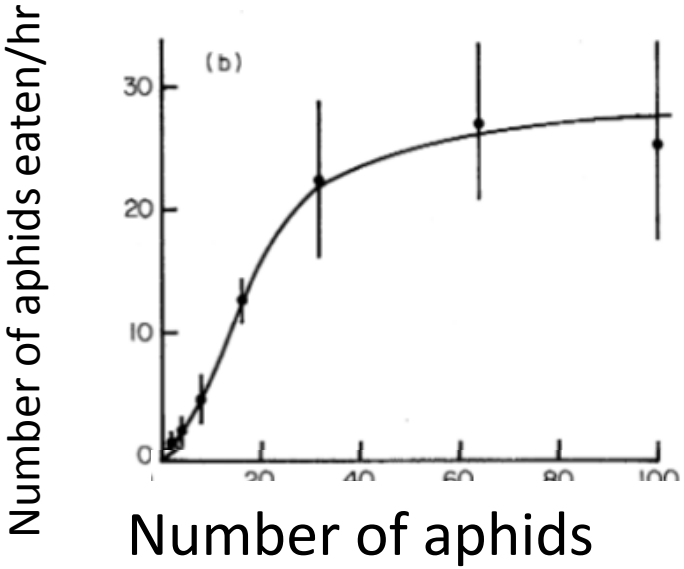
Aphid and ladybugs

Let x = population size of aphids.

Let y = populaiton of ladybugs

G(x) =Aphids population growth rate

P(x,y)=Aphid predation rate by ladybugs.



Aphid and ladybugs

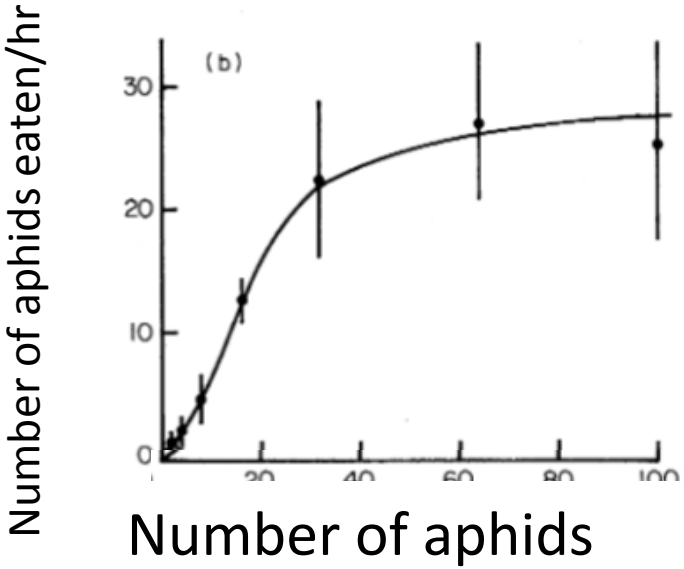
Let x = population size of aphids.

Let y = populaiton of ladybugs

G(x) =Aphids population growth rate = r x

P(x,y)=Aphid predation rate by ladybugs

$$P(x,y) = y \cdot \left(\frac{30x^2}{20^2 + x^2} \right)$$



Previously: Aphid and ladybugs

When does growth rate balance predation rate?

$G(x)=P(x)$

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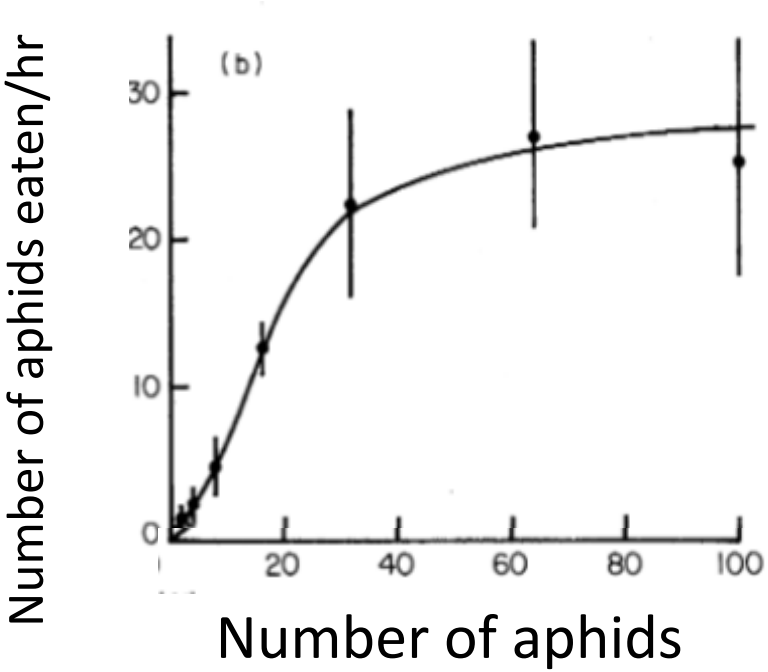


Now:

What happens to the aphid population when these do NOT match!

The aphid population will change!

We can use a differential equations to track that change

Rate of change of aphid population

$$\frac{dx}{dt} = \text{Growth rate} - \text{predation rate}$$

$$= G(x) - P(x,y)$$

Net growth rate

$F(x,y)$

We can determine how the aphid population changes by drawing a slope field or state-space diagram of the equation $dx/dt = F(x)$

Example: 1 ladybug (y=1)

Predation rate= P(x,y) = $K\frac{x^2}{a^2+x^2}$

Growth rate = G(x) = rx

Net growth rate F(x) = G(x) − P(x)

= $rx - K\frac{x^2}{a^2+x^2}$

Problem is now:

Sketch the qualitative behaviour for the differential equation

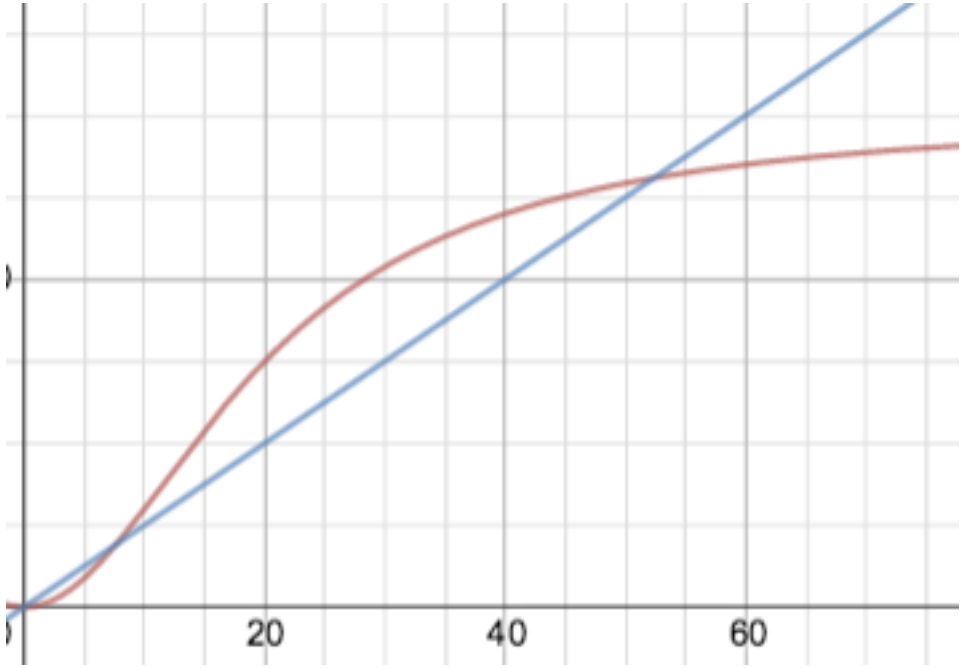
$$\frac{dx}{dt} = \underbrace{rx - K\frac{x^2}{a^2 + x^2}}$$

$$= \quad F(x)$$

Predation and growth rate:

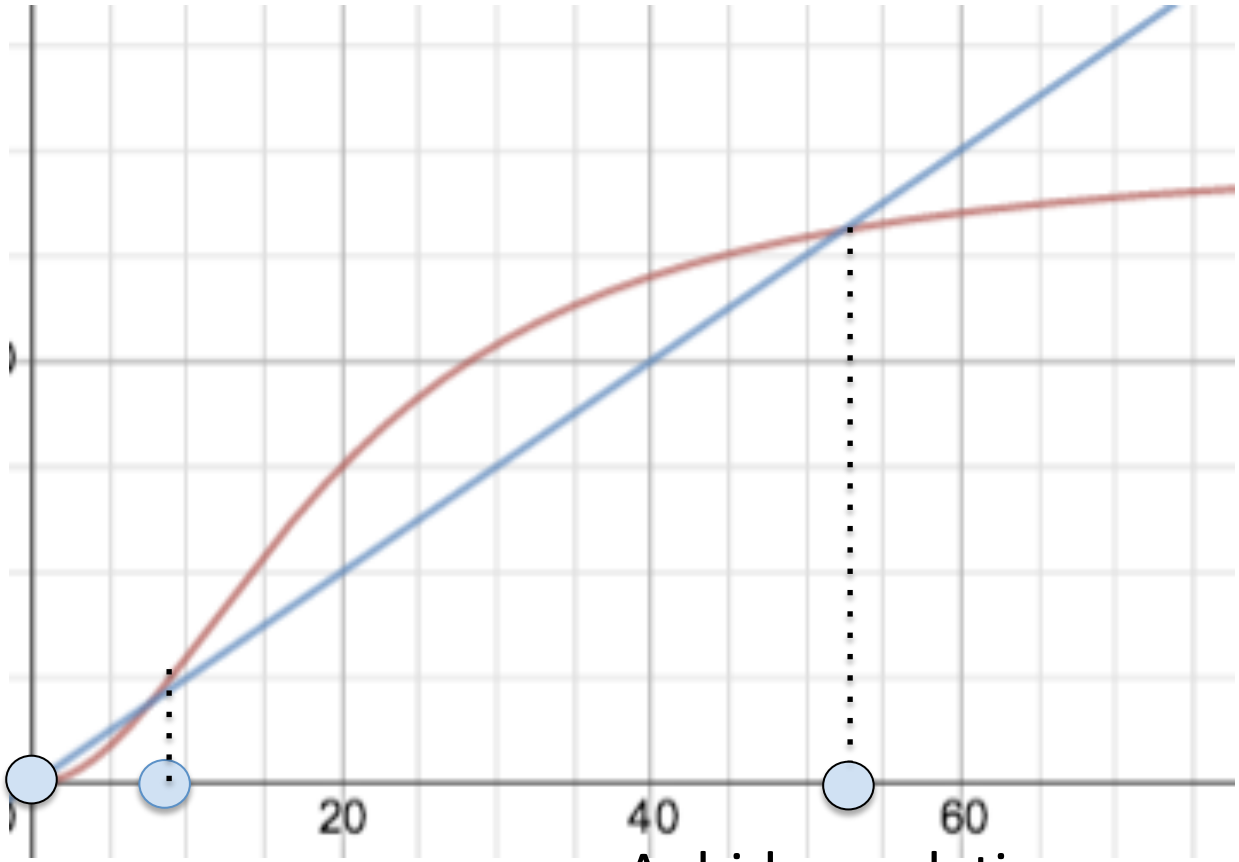
$$P(x) = K\frac{x^2}{a^2 + x^2}$$

$$G(x) = rx$$



Steady states:

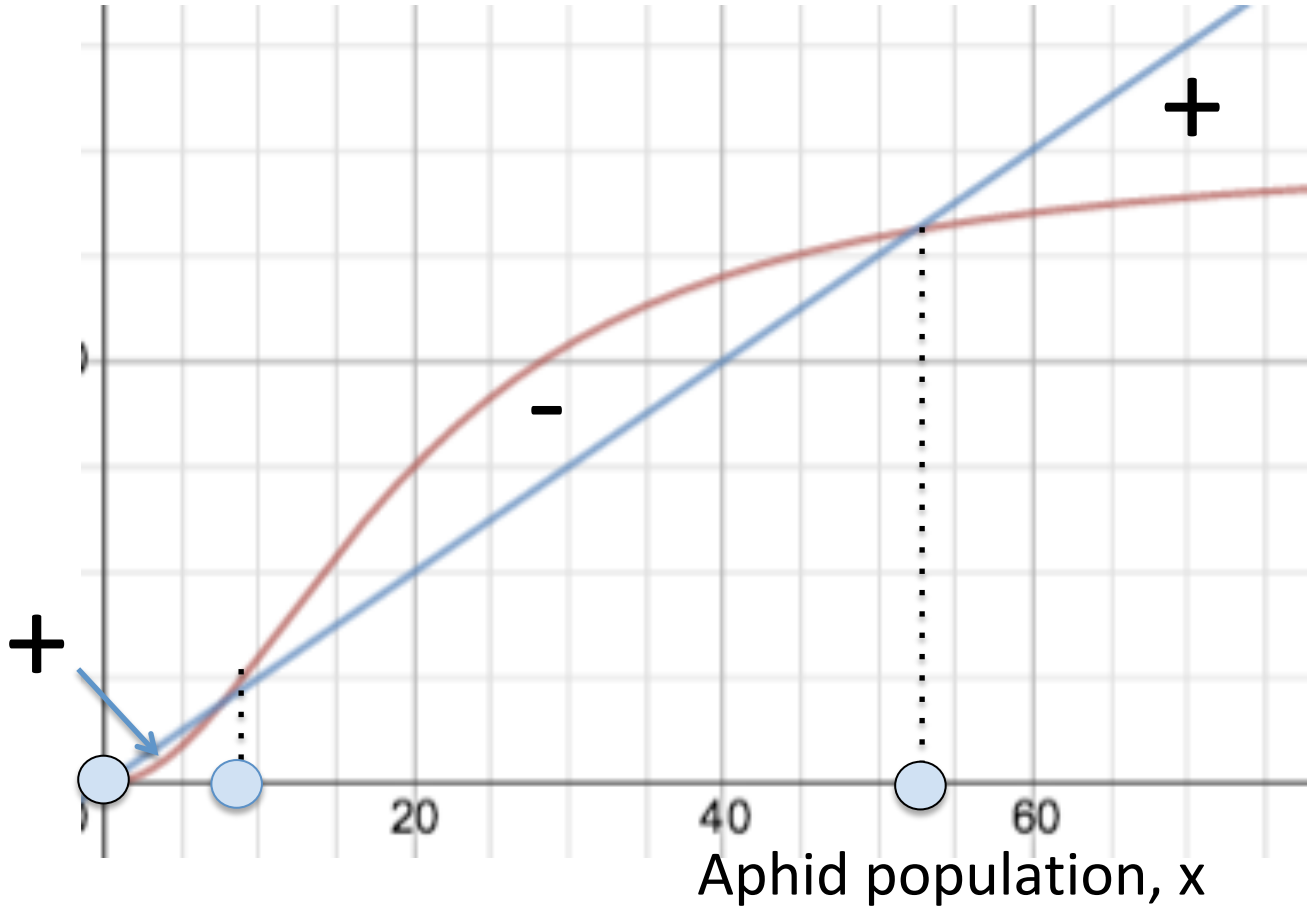
Aphid population does not change if G(x) =P(x)



Predictions:

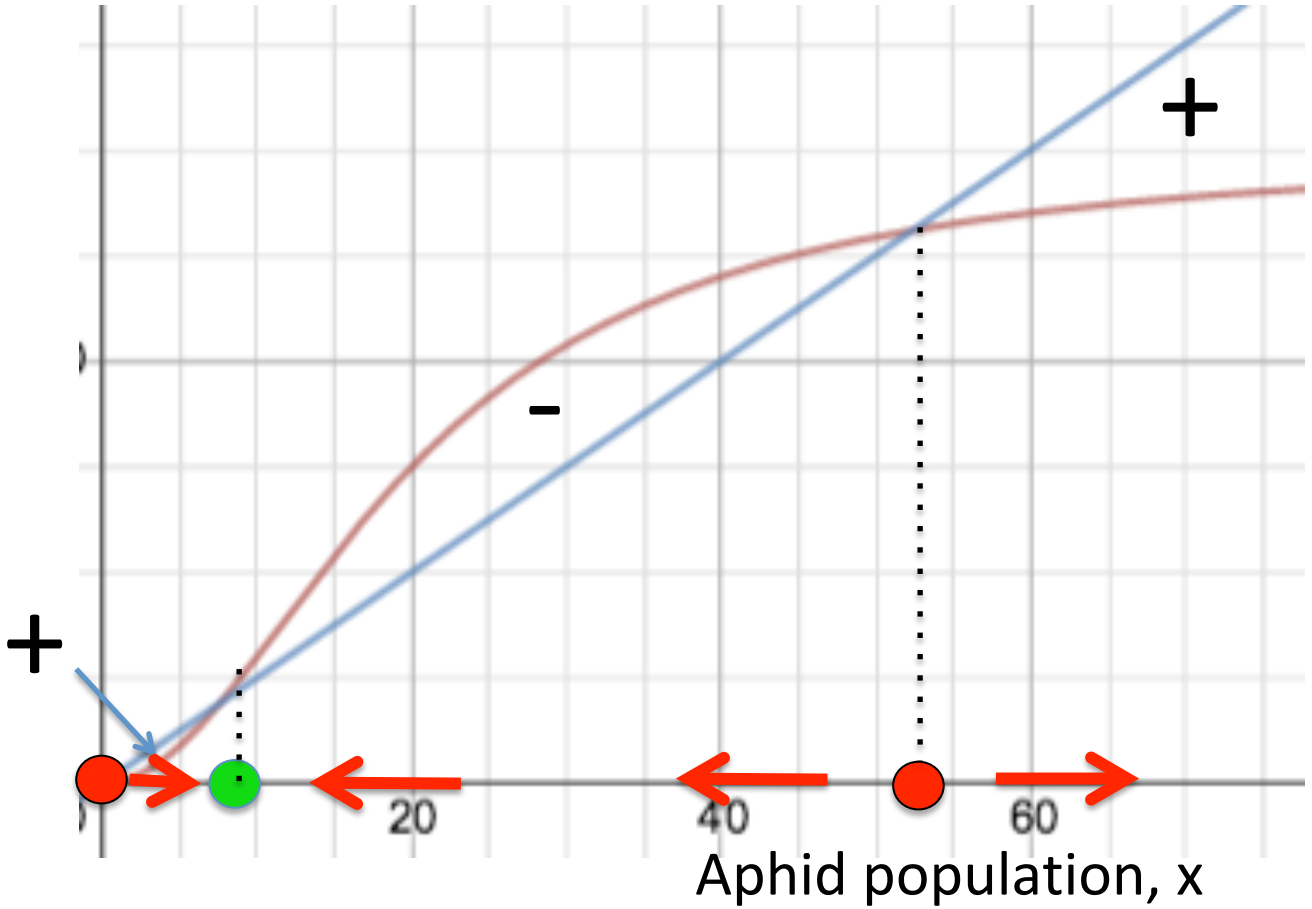
Aphid population increases if G(x) > P(x)

Aphid population decreases if G(x) < P(x)



Directions of flow:

The sign of G(x) - P(x) indicates if x increases or decreases



Stability of steady states

Unstable Stable Unstable

Altrenatively:

- Sketch the function F(x)

- $$\frac{dx}{dt} = \underbrace{rx - K\frac{x^2}{a^2 + x^2}}$$

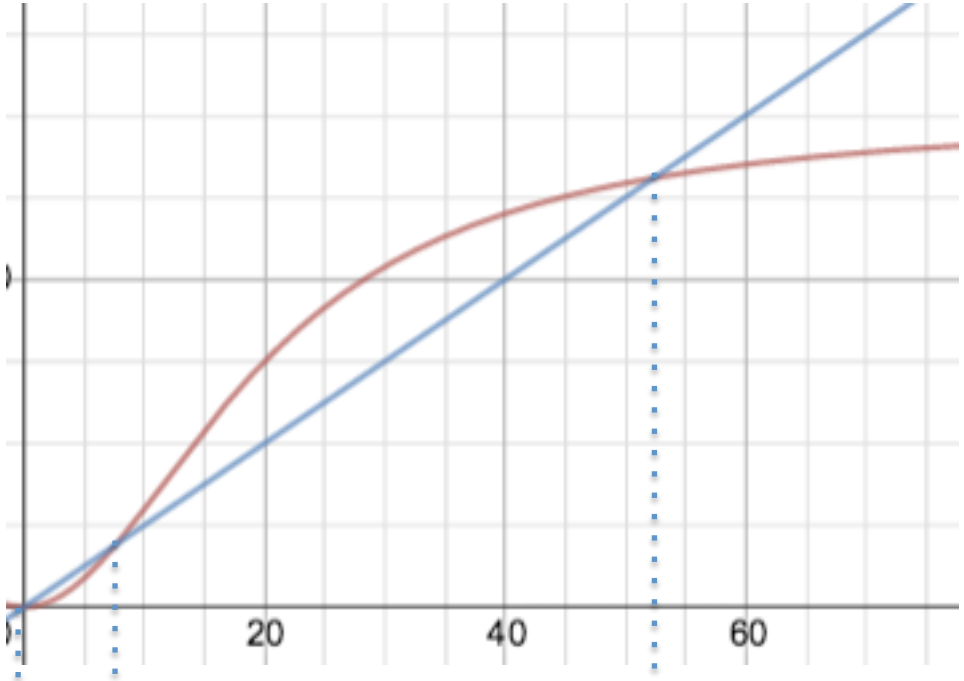
  $= \quad F(x)$

F(x)=G(x)-P(x)

- Both functions:

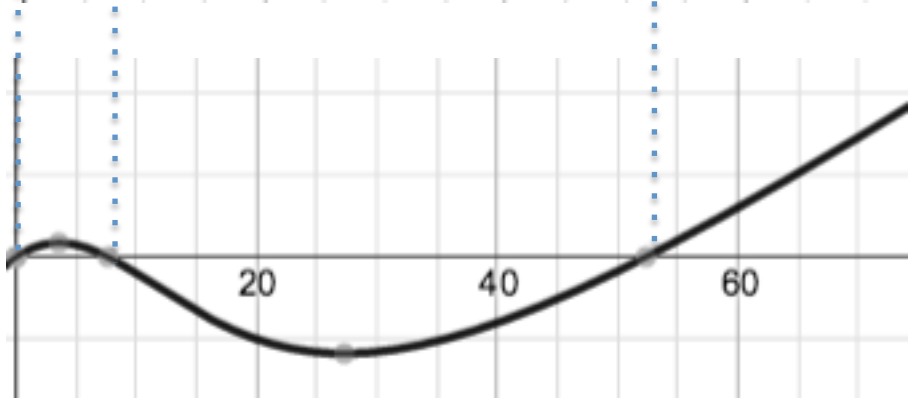
$$P(x) = K\frac{x^2}{a^2 + x^2}$$

$$G(x) = rx$$

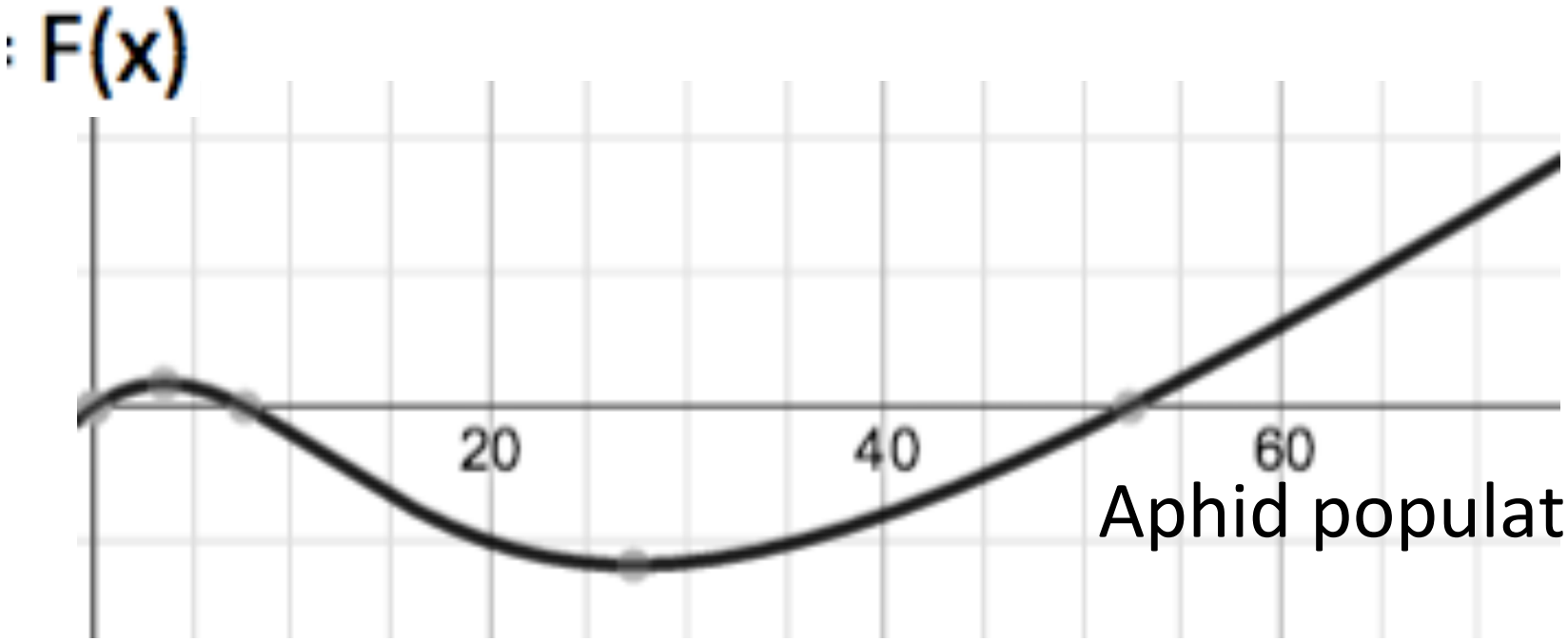


- Subtracting:

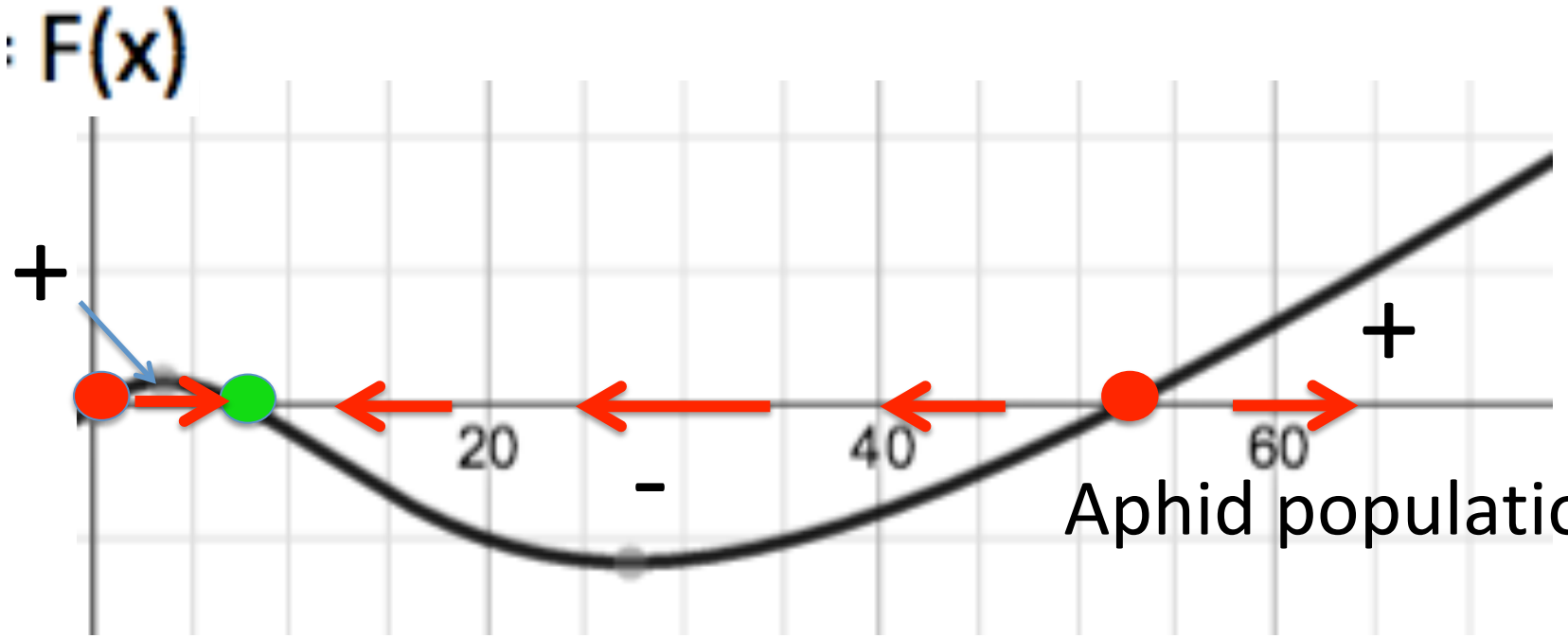
$$F(x) = G(x) - P(x)$$



Net growth rate



State-space diagram:



What does this tell us?

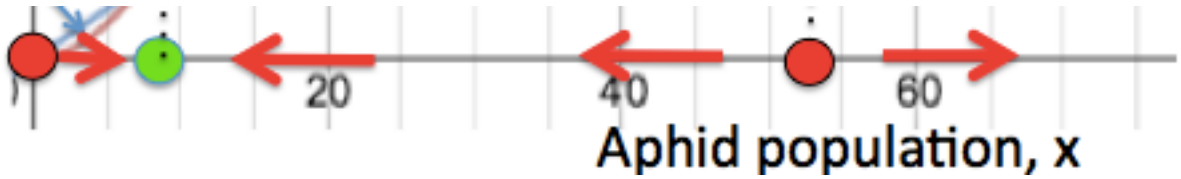
(A) One ladybug can control the aphid population

(B) The aphid population will always grow

(C) The ladybug population has a carrying capacity

(D) If there are too many aphids initially, then one ladybug will not be able to control their level

(E) A large aphid population cannot be sustained due to resource limitations.



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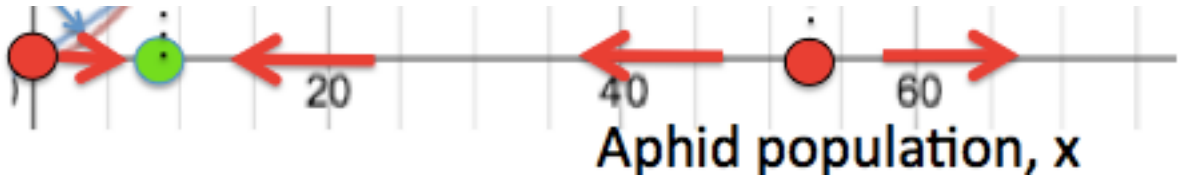


For further study:

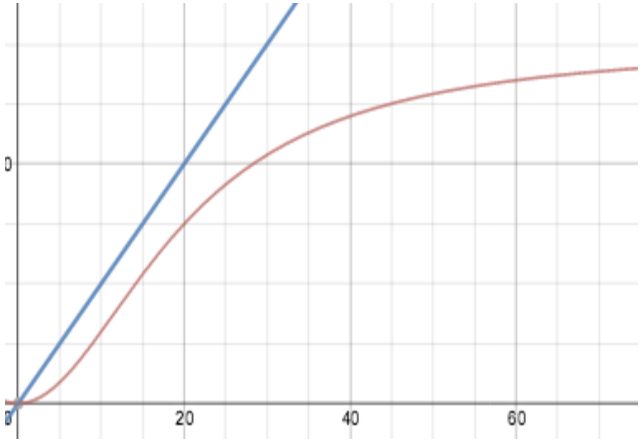
(1) Draw a slope field and some solution curves for the same system

(2) How would your conclusions change for a type II predator response?

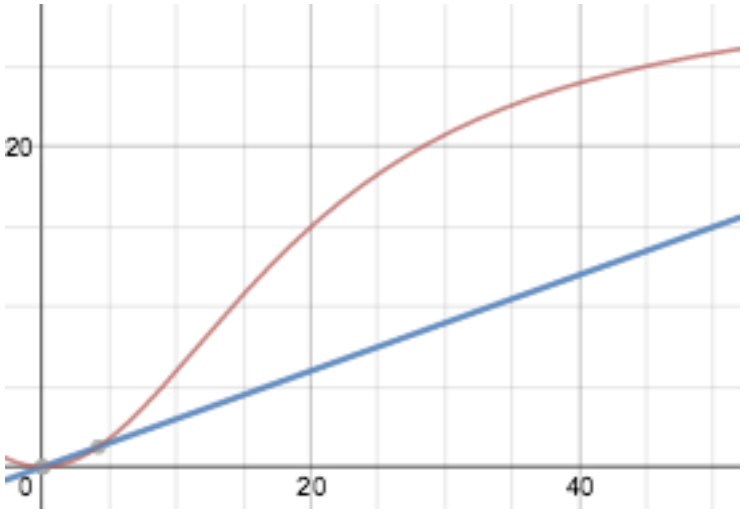
What if..

The growth rate of aphids is really high (1) or really low (2)?

(1)

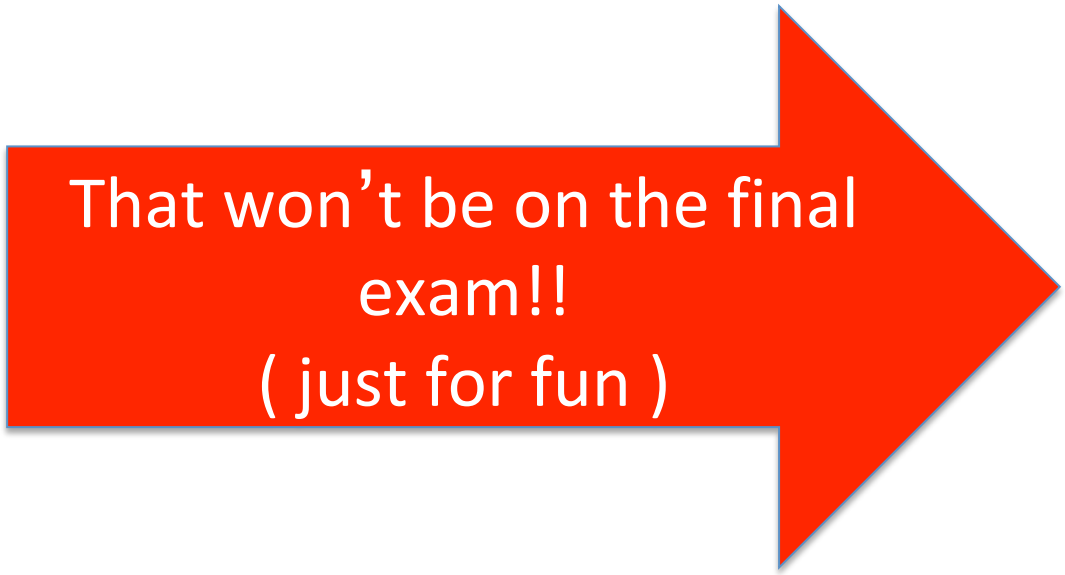


(2)



And now..

- Here is some cool stuff



What if the predator population also changes?

Aphids: dx/dt = rate of (birth –mortality)
-rate of predation

Ladybugs: dy/dt = rate of birth – rate of mortality

What if the predator population also changes?

Aphids: dx/dt = rate of (birth –mortality)
-rate of predation

The more I eat, the more babies I make

Ladybugs: dy/dt = rate of birth – rate of mortality

What if the predator population also changes?

Aphids: $dx/dt = r\,x - P(x, y)$

Ladybugs: $dy/dt = k\,P(x, y) - d\,y$

The populations are linked to one another!

"Coupled system of differential equations"

Keep track of BOTH populations

Aphids: $dx/dt = r\,x - P(x, y)$

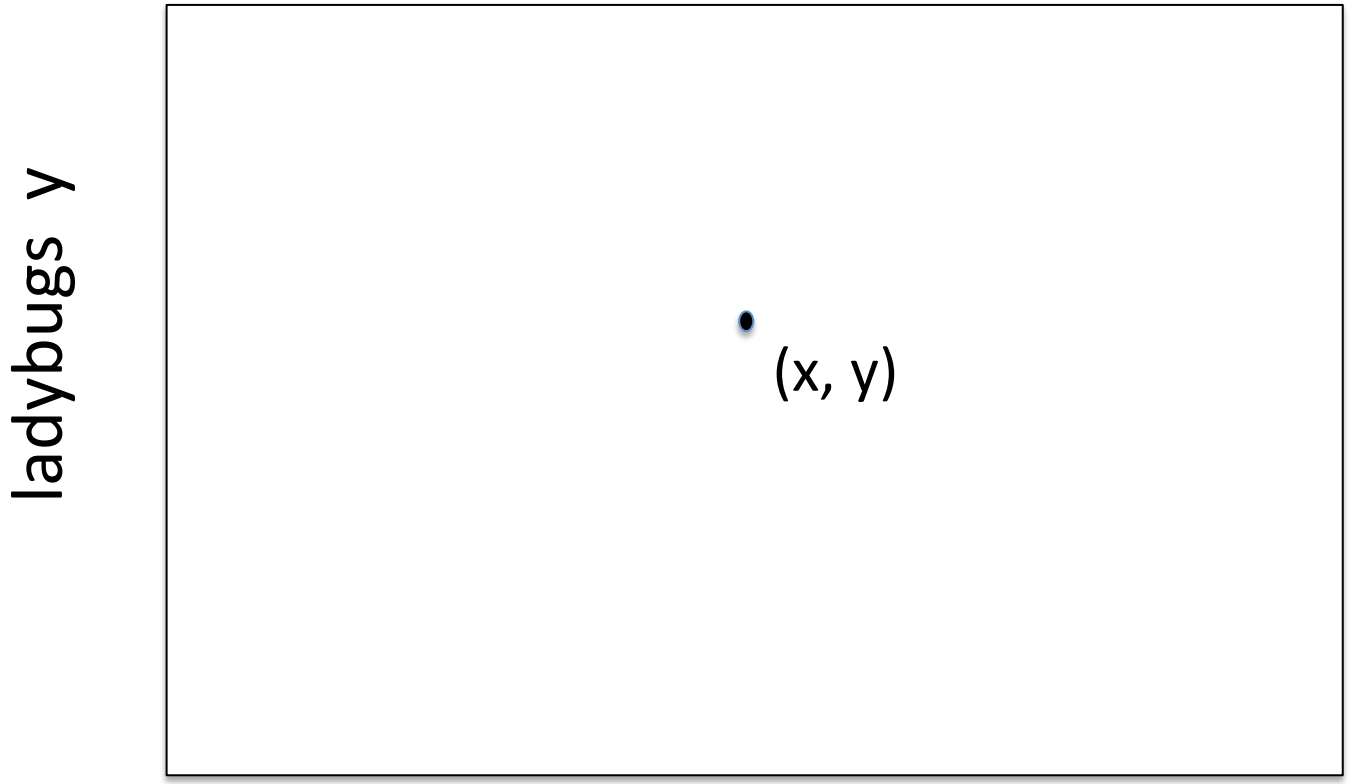
Ladybugs: $dy/dt = k\,P(x, y) - d\,y$

We can still get an idea of what happens using the concept of slope field.

Constructing a slope field

$dx/dt = r\,x - P(x,y)$
$dy/dt = k\,P(x, y) - dy$

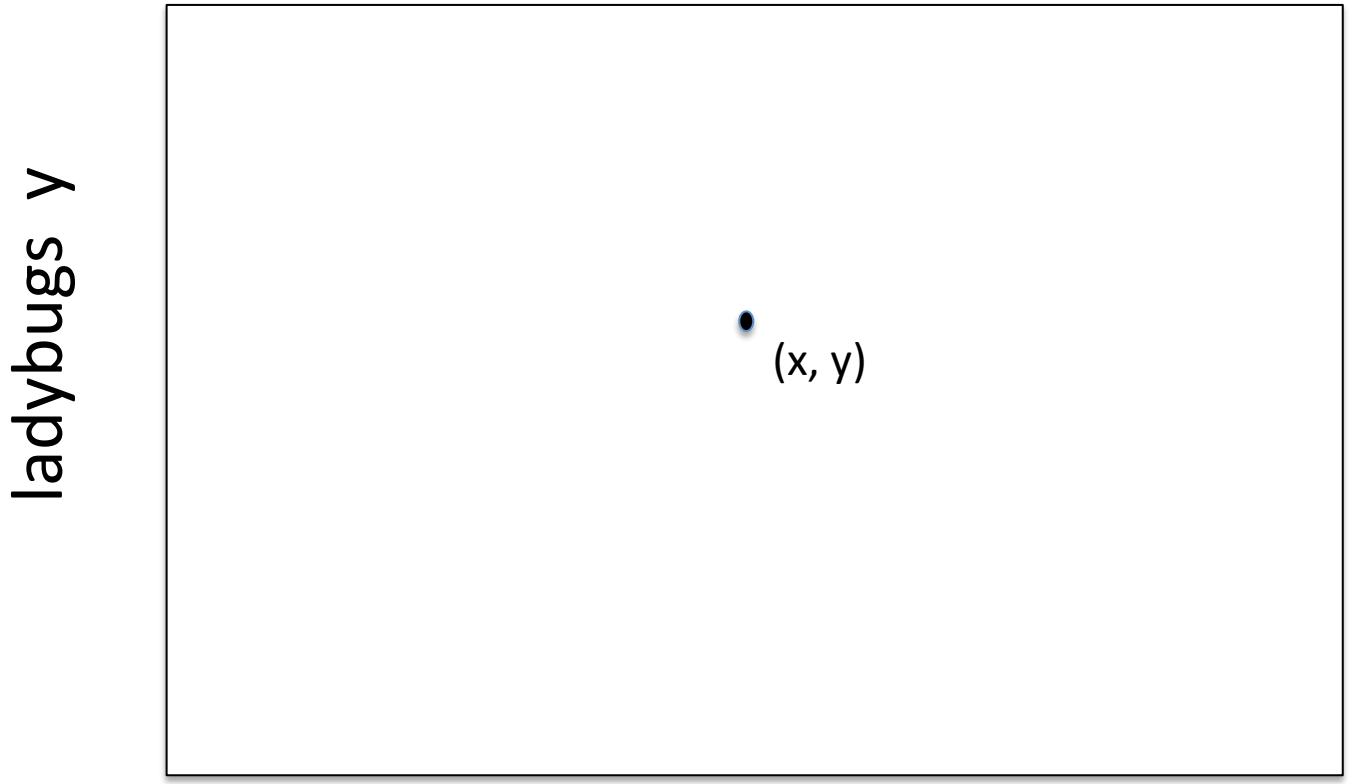
Pick any point:



$dx/dt = r\,x - P(x,y)$
$dy/dt = k\,P(x, y) - dy$

Keep track of both x and y

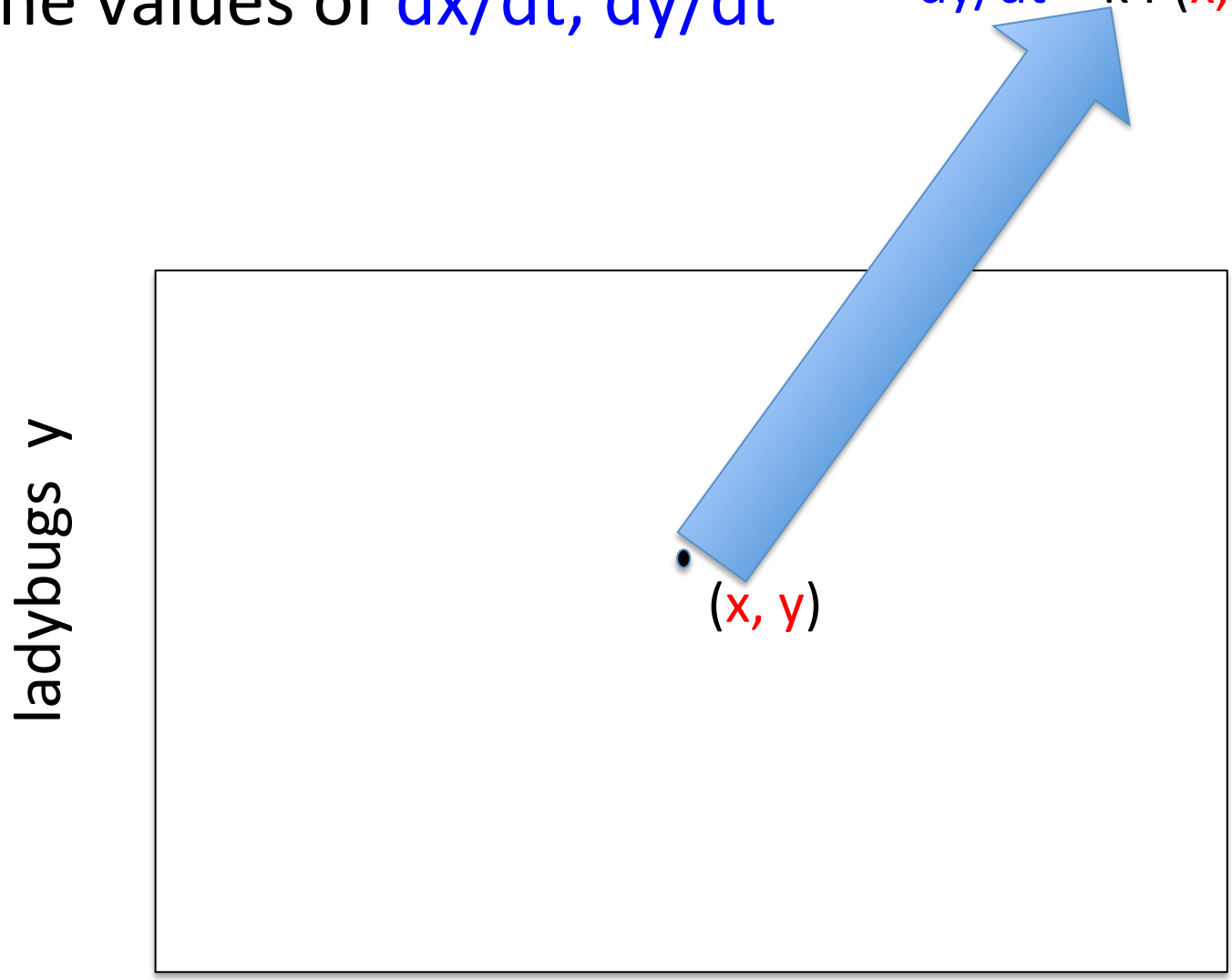
Pick a point with known x, y coord:



Determine values of dx/dt, dy/dt

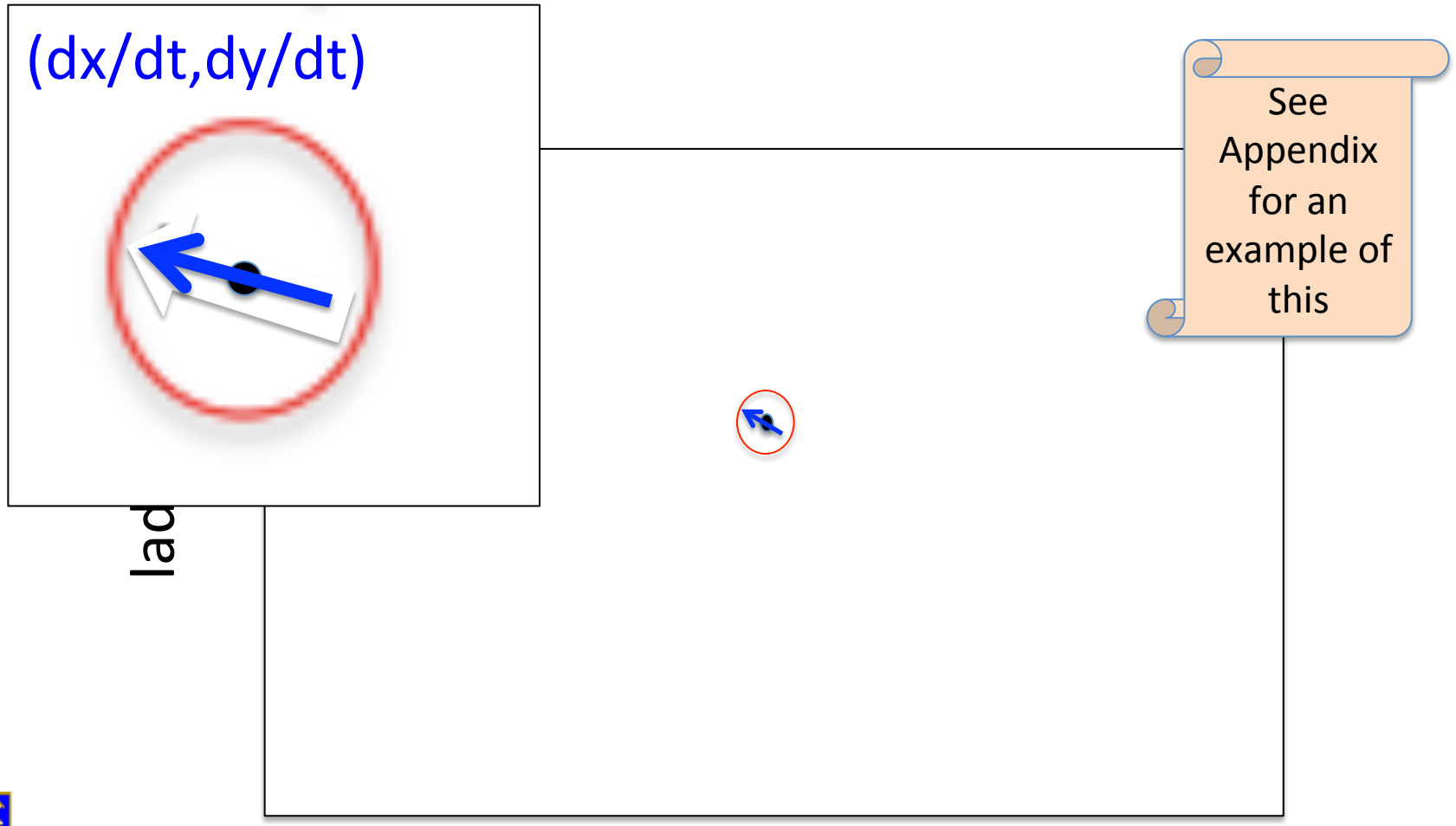
dx/dt = r x – P(x,y)

dy/dt = k P(x, y) – dy



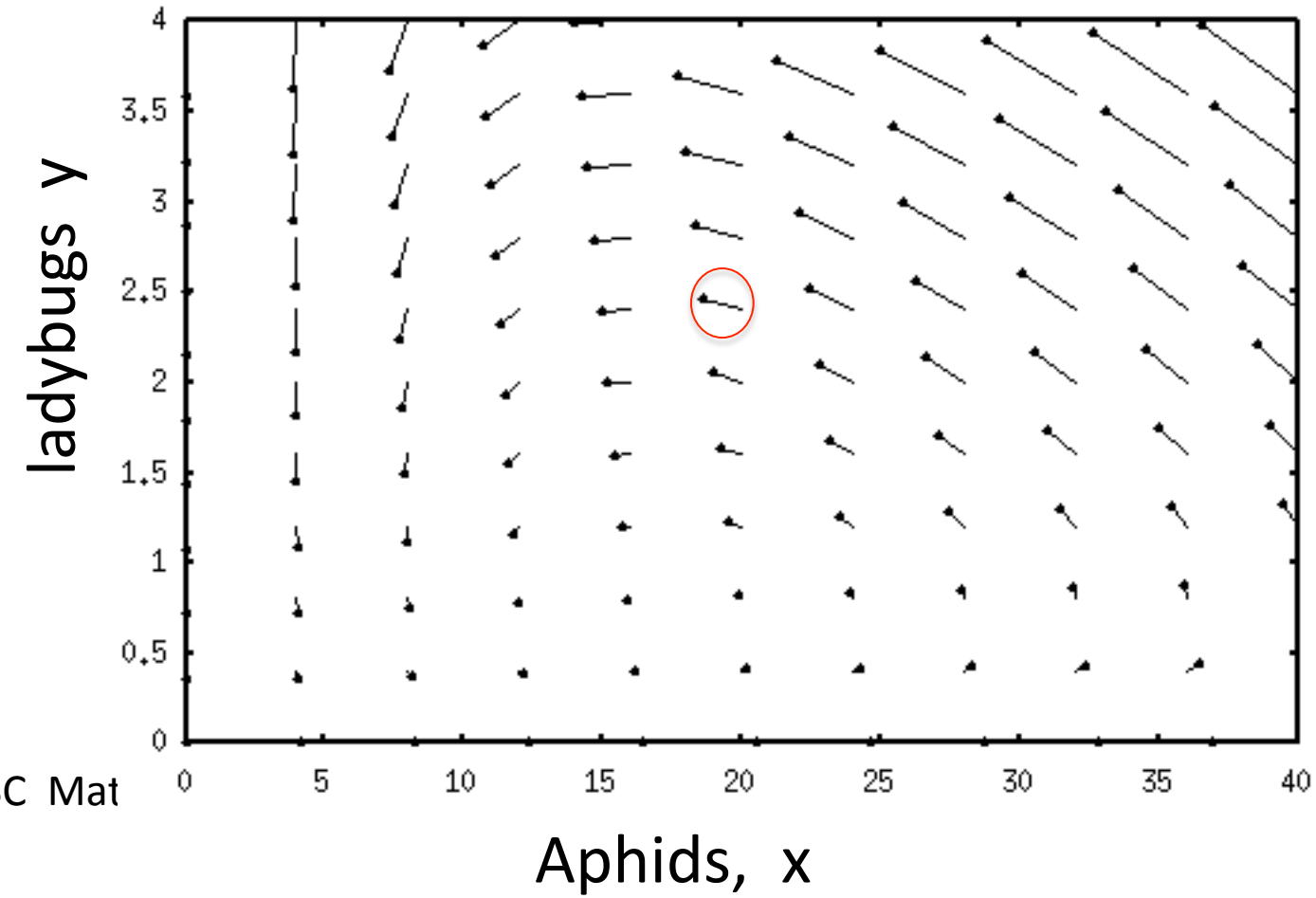
Draw an arrow representing changing x and y at this point

$dx/dt = r\,x - P(x,y)$
$dy/dt = k\,P(x, y) - dy$



$dx/dt = r\,x - P(x,y)$
$dy/dt = k\,P(x, y) - dy$

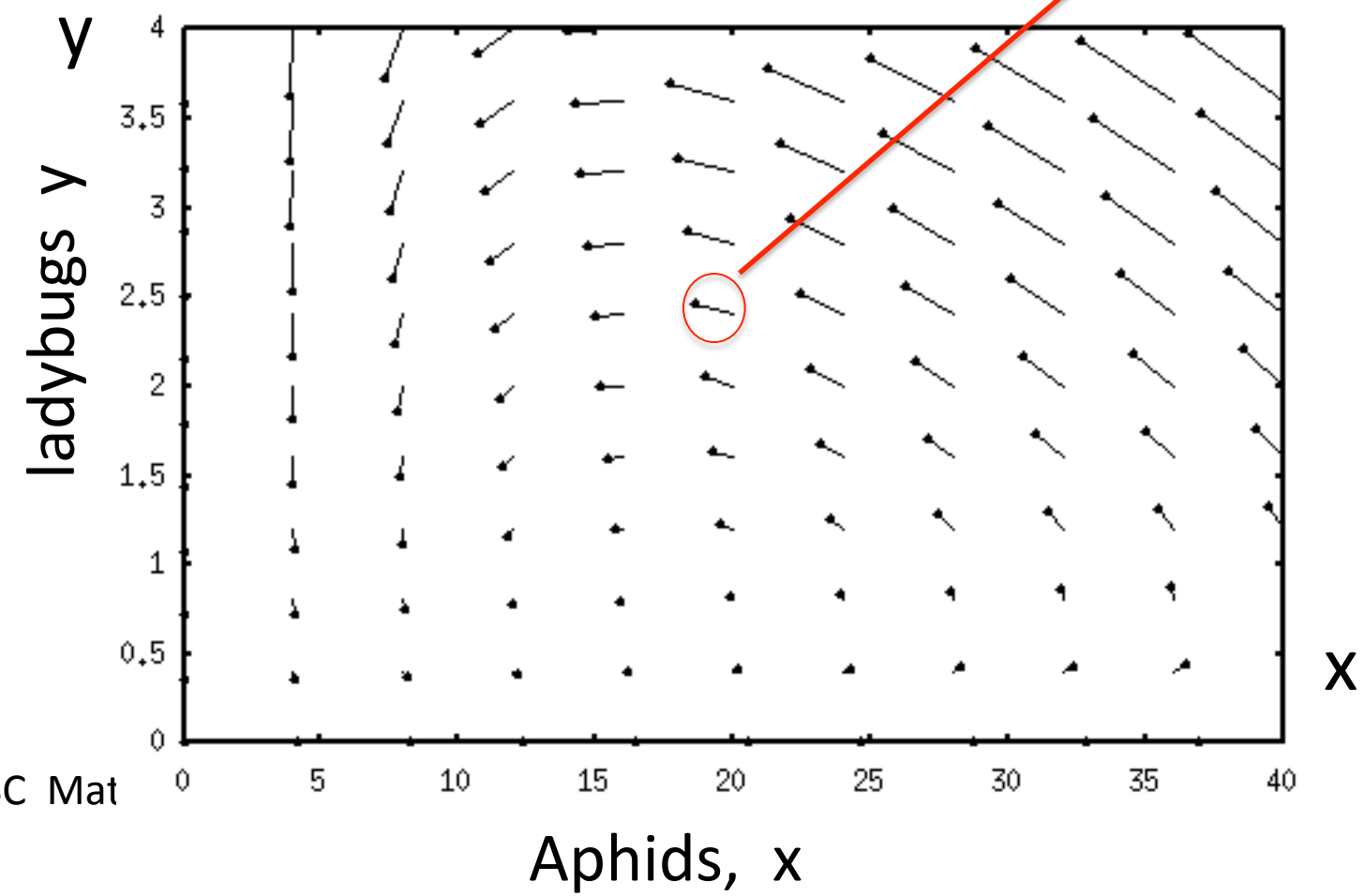
Continue this at many points:



$$\frac{dx}{dt} = r\,x - P(x)\,y$$
$$\frac{dy}{dt} = k\,P(x)\,y - d\,y$$

Direction field

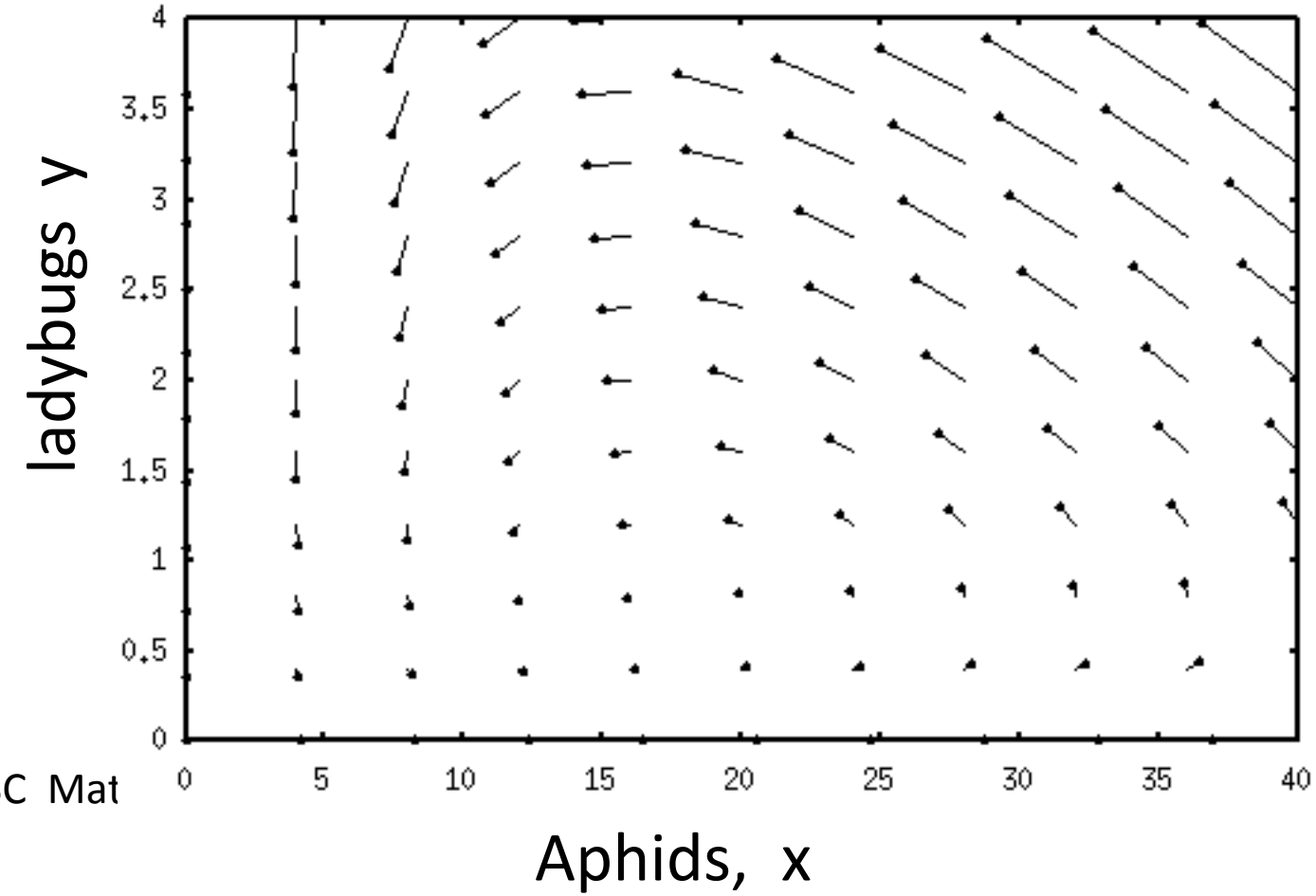
(dx/dt,dy/dt)



UBC Mat

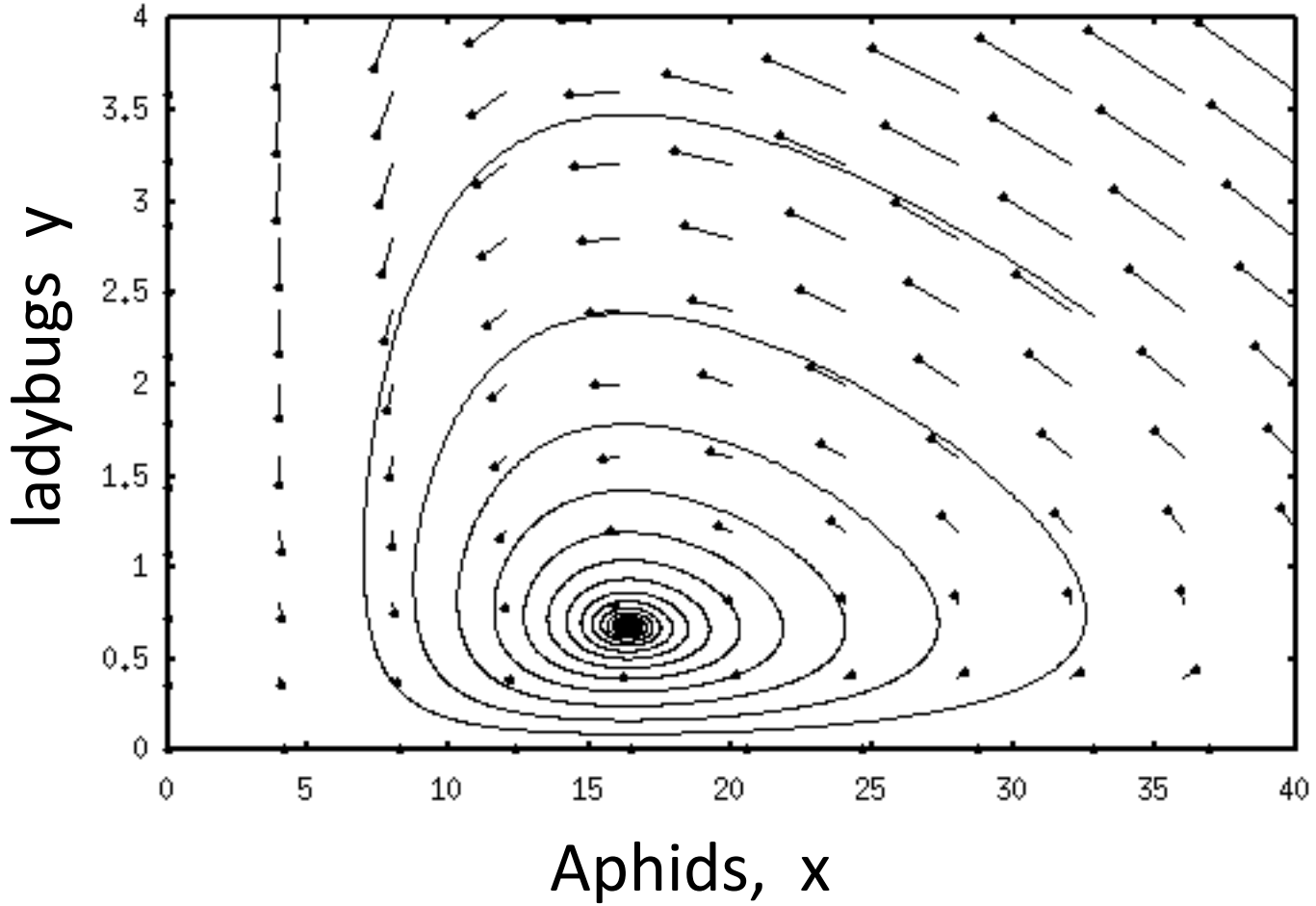
$dx/dt = r\,x - P(x,y)$
$dy/dt = k\,P(x, y) - dy$

Using the direction field, draw solution curves:



$dx/dt = r\,x - P(x)\,y$
$dy/dt = k\,P(x)\,y - d\,y$

Solution curve



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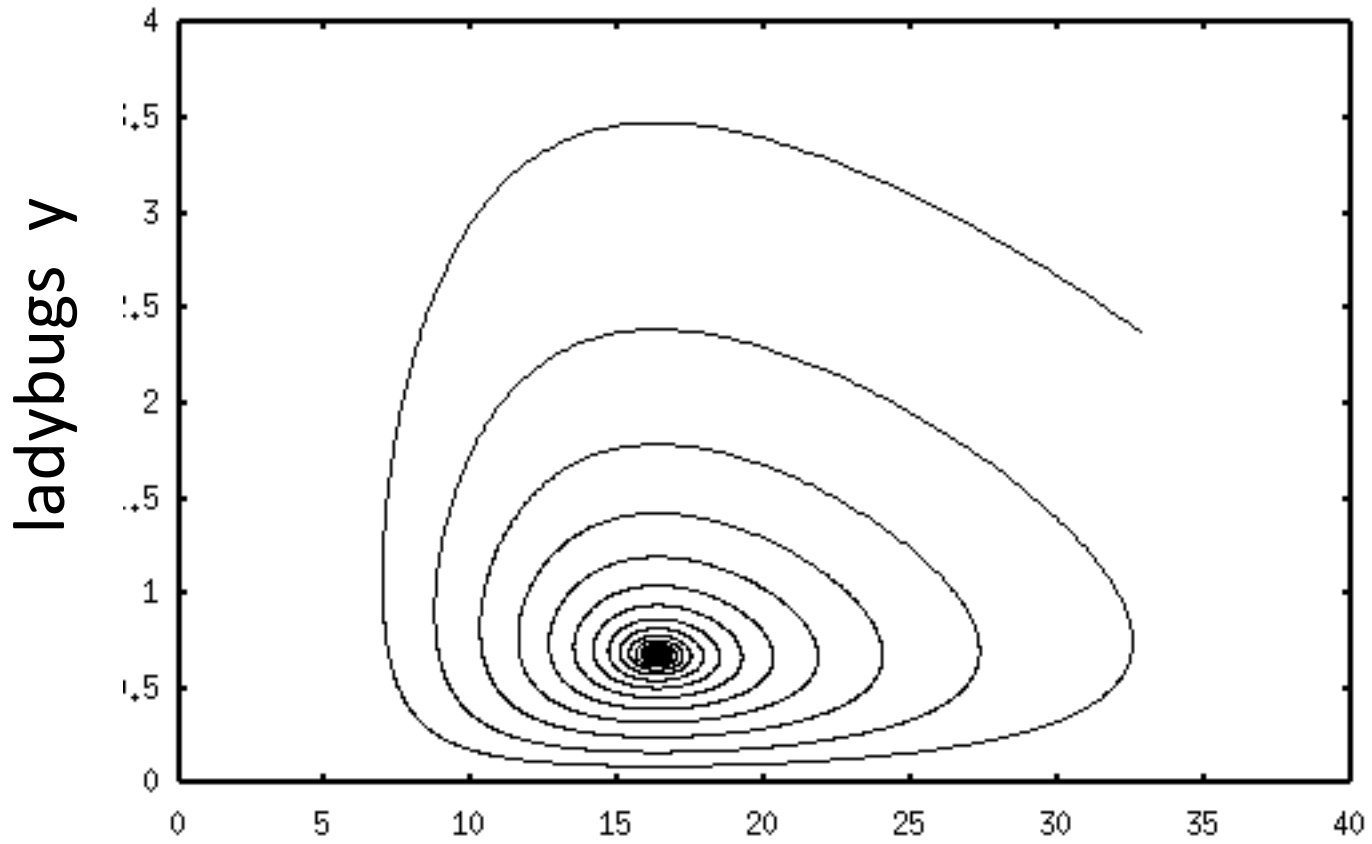
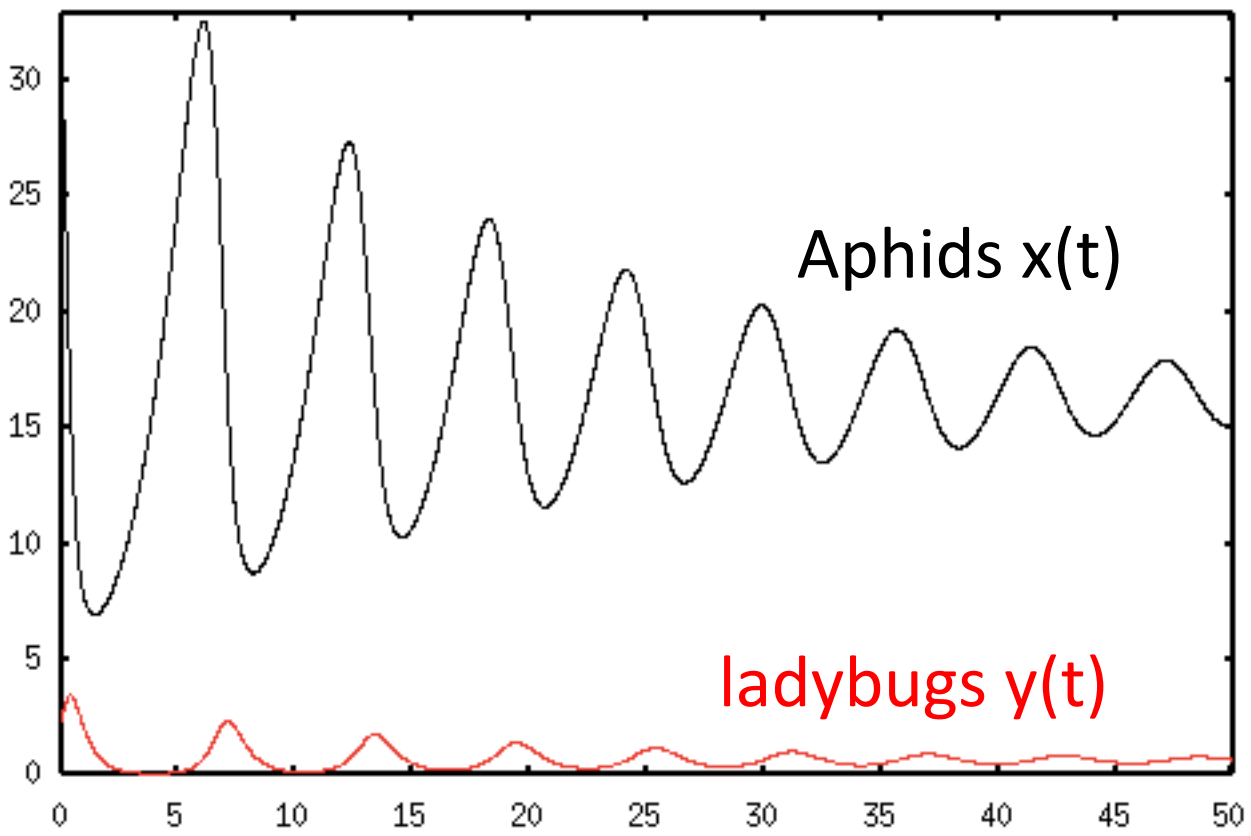
$dy/dt = k\,P(x)\,y - d\,y$

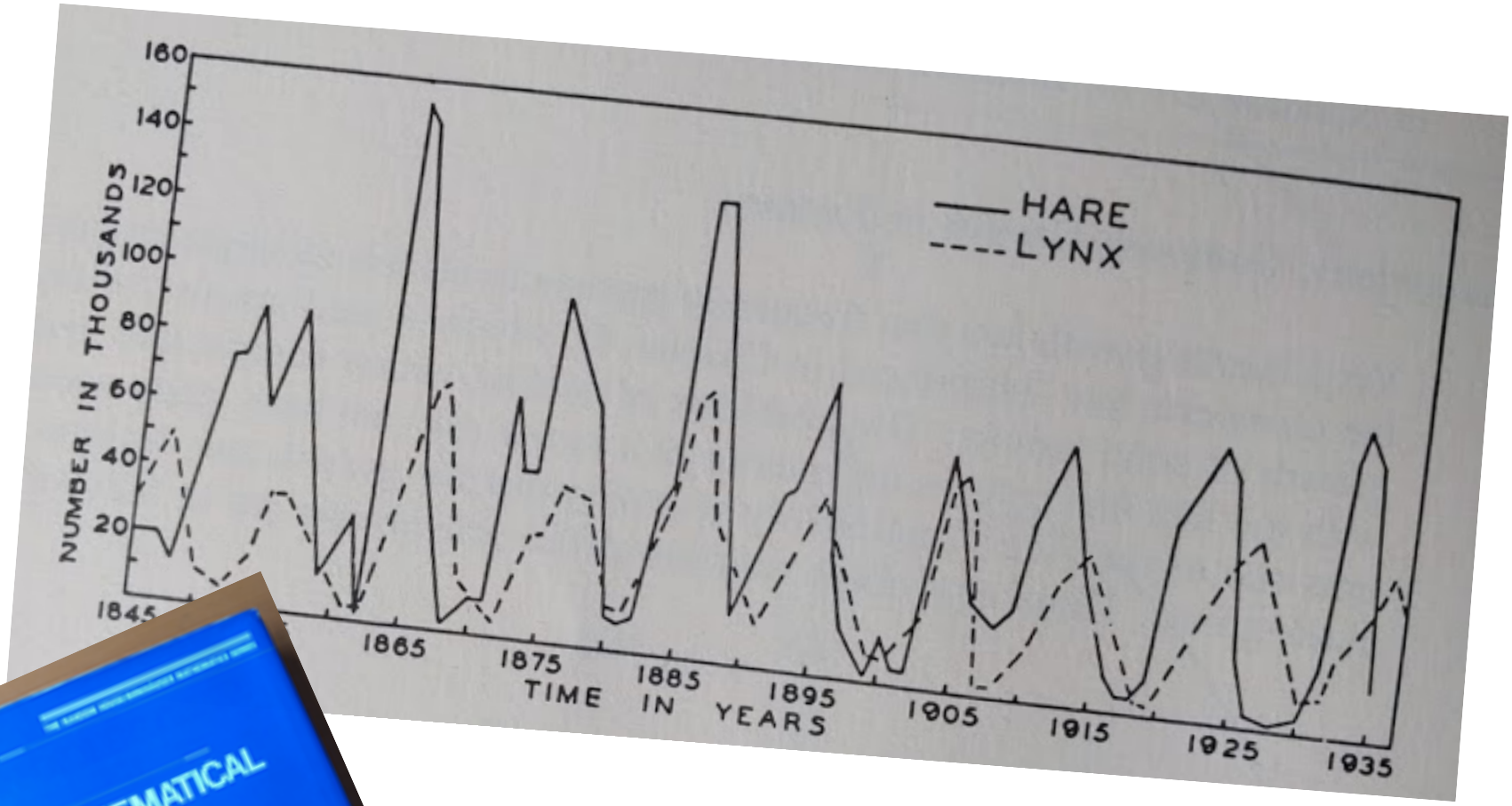
Solution curve

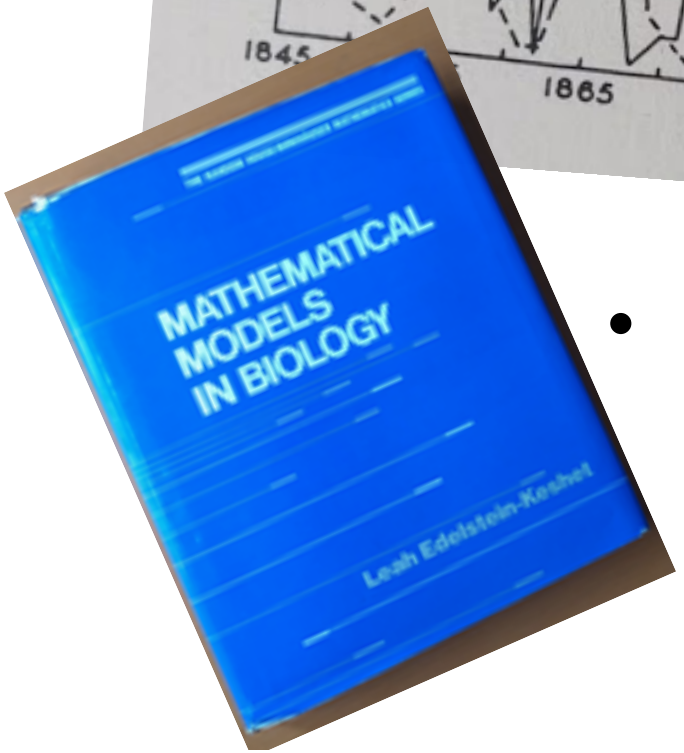


Now add a time axis

Cycles of aphids and ladybugs!



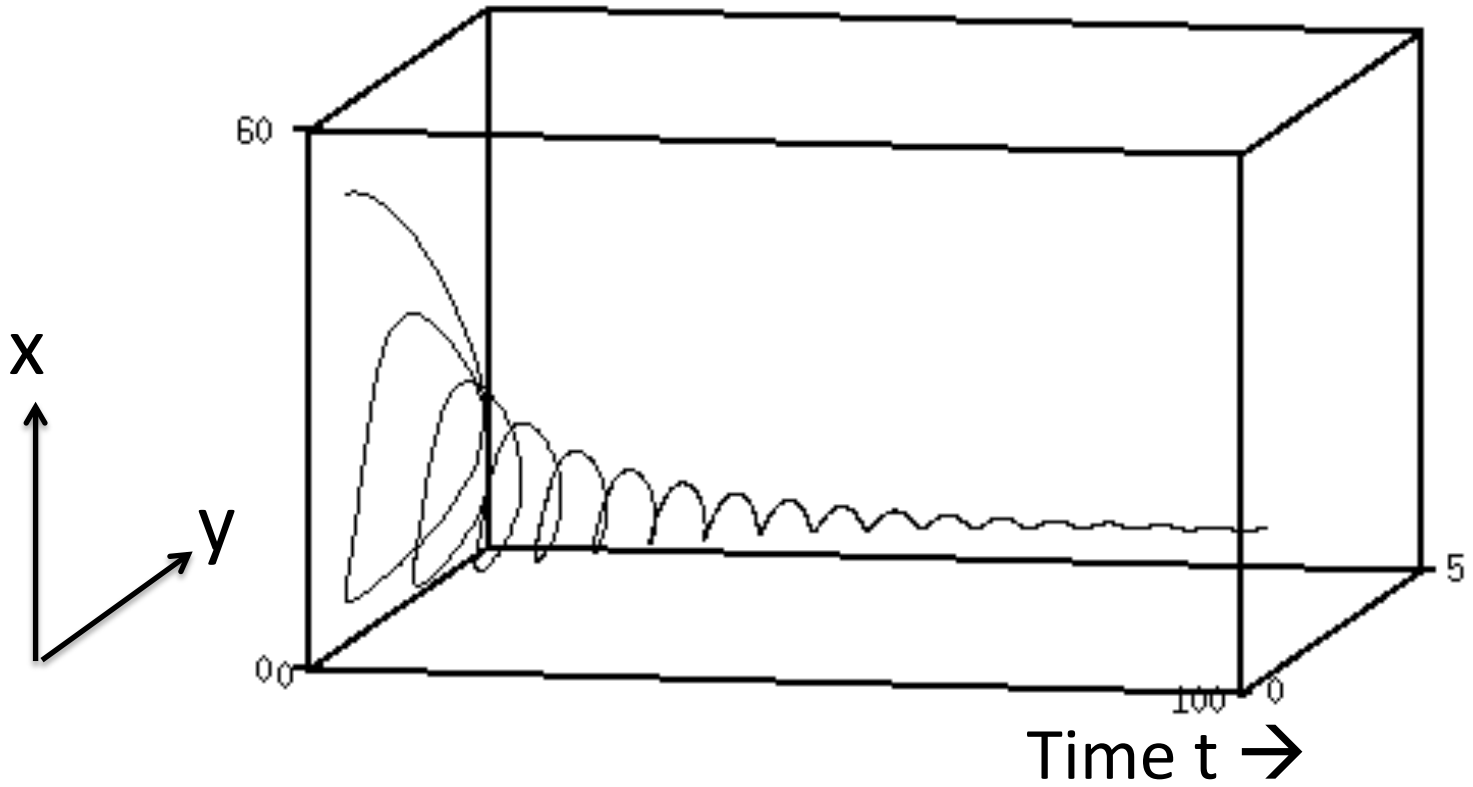




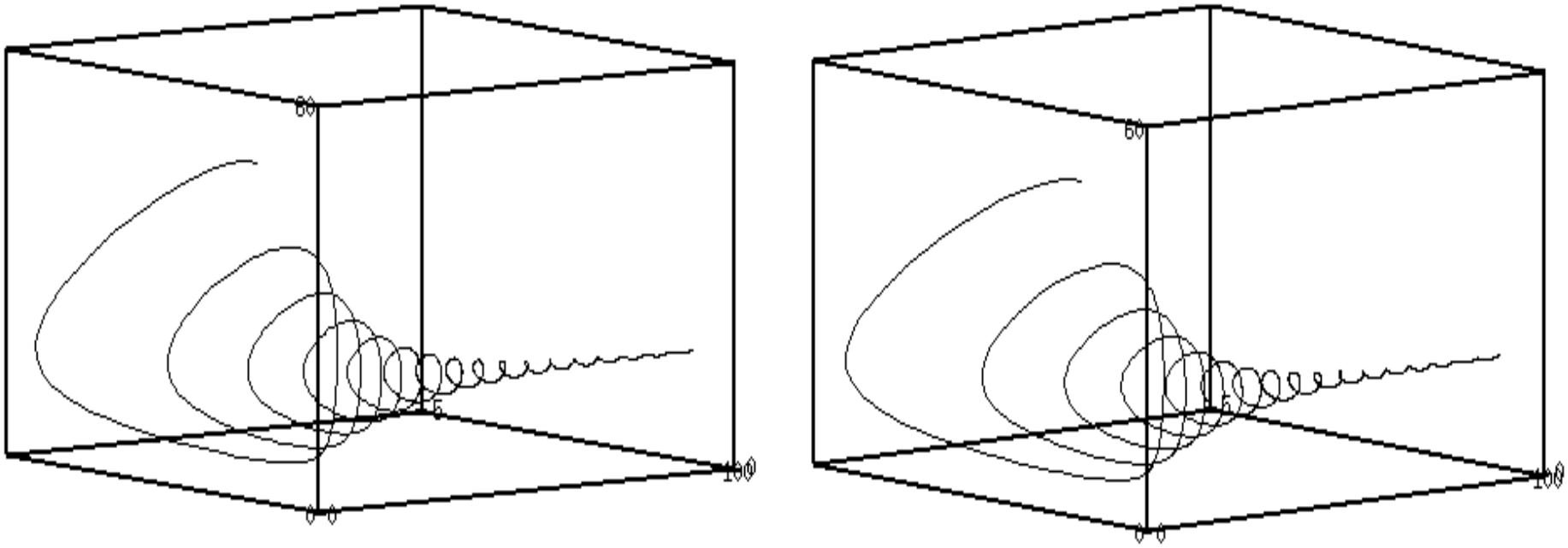
- Real cycles of predators and prey (lynx and hare)

The solution curves are now in 3D

Curve for (t,x(t),y(t)):



Stereoscopic view



Now preparation for quiz 3

- But first, a reminder of a differential equation we saw a while ago.

Flow of fluid out of cylinder

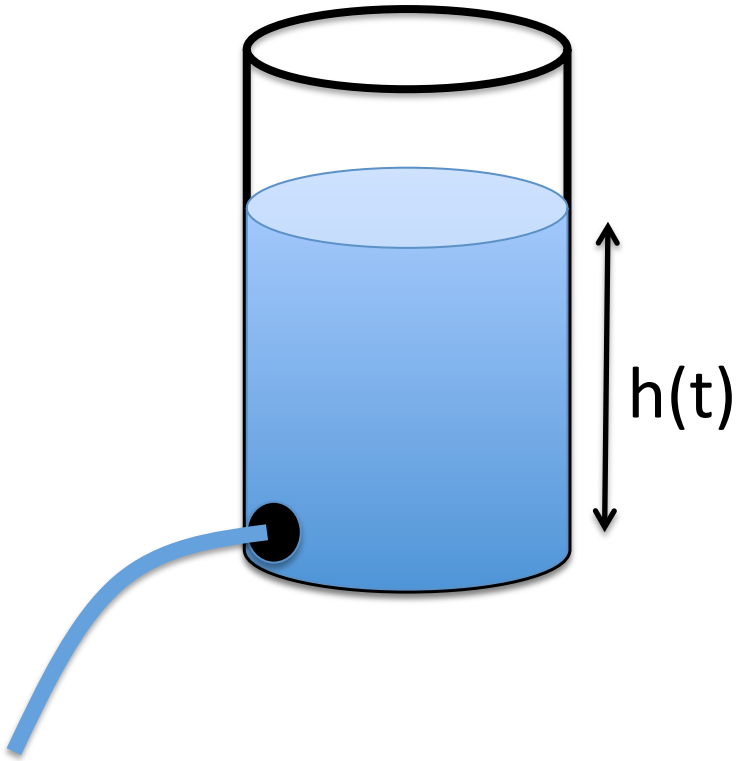
Cylindrical tank of fluid.

Fluid height h(t)

DE: $$\frac{dh}{dt} = -k\sqrt{h},$$

with $$h(0) = h_0$$

And $$k = \frac{a}{A}\sqrt{2g}$$



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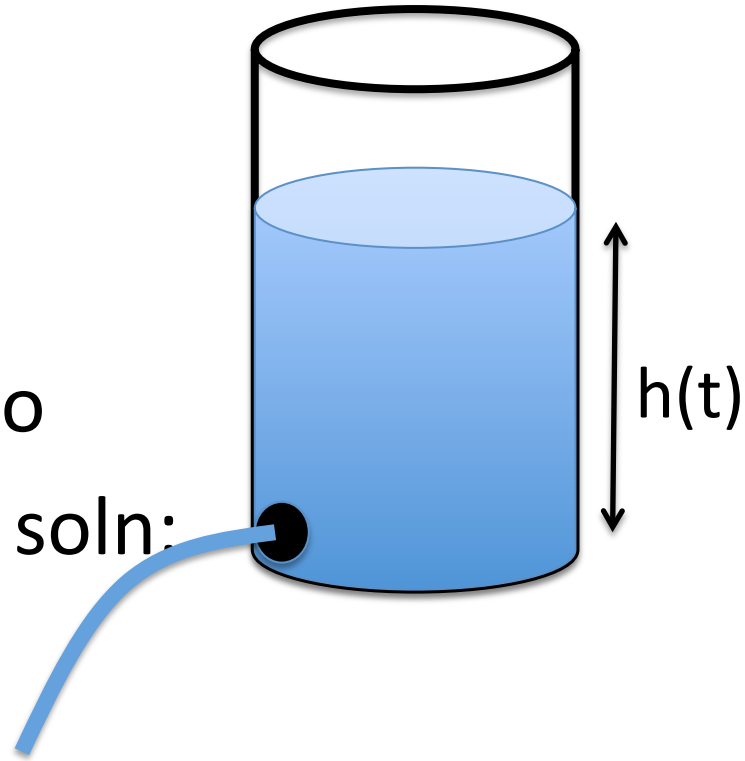


Last week we showed how to check that the following is a soln:

$$h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2$$

How good is this prediction?

- I was wondering if this theoretical result has any bearing on reality..
- Let us do an experiment..

Data (from experiment)

- In the group-version of the quiz, I'll ask you to see if this data fits the model. This is meant to be a learning activity (for which you get some bonus points).

Data (from my own home expt)

time t (sec)	height (cm) h
5	12
25	10
39	8
57	6
79	4
104	2

Data from experiment in class

Fluid height (cm)	Time (sec)
14	0
12	13.46
10	26.33
8	40.86
6	58.35
4	79
2	106

Quiz 1

Individual stage

- No notes
- No books
- No computers or calculators

- OK to use scrap paper

Quiz 1

- Q1

Consider a cubic crystal of side length x and volume $V = x^3$. If the volume is increasing at the rate of 3 mm^3 per month, and the crystal remains a cube, at what rate is its side length changing (in mm/month) when its volume is 8 mm^3?

(A) $3^{1/3}$, (B) $\frac{3^{1/3}}{8}$, (C) $\frac{9}{64}$, (D) $\frac{1}{64}$, (E) $\frac{1}{4}$.

Quiz 1

- Q2

The human population on Earth doubles roughly once every 50 years. If there are 7.5 billion people on Earth now, in how many years will there be 30 billion?

(A) $7.5e^{50t}$ years (B) $7.5e^{t/50}$ years (C) $\frac{7.5}{30}e^{t\ln(2)/50}$ years

(D) 75 years (E) 100 years

Quiz 1

- Q3

Consider the function $y = ax^b$. When I graph $\ln(y)$ versus $\ln(x)$, I see a line with slope $3/4$ and vertical intercept 1. This tells me that

(A) $a = 1, b = 3/4$ (B) $a = 3/4, b = 1$ (C) $a = e, b = 3/4,$

(D) $a = 3/4, b = e$ (E) $a = 1, b = e^{\frac{3}{4}}$.

Quiz 1

- Q4

Which function is a solution to the differential equation $\frac{dh}{dt} = -k\sqrt{h}$ with $h(0) = h_0$?

(A) $h(t) = (\sqrt{h_0} - \frac{1}{2}kt)^2$

(B) $h(t) = (\sqrt{h_0} - kt^2)$

(C) $h(t) = h_0 e^{-kt}$

(D) $h(t) = (1 - e^{-kt})$

(E) None of the above

Quiz 1

- Q5

Which of the following differential equations has unstable steady states at $y = \pm 1$ and a stable steady state at $y = 0$? (Hint: consider sketching the state-space diagrams).

(A) $\frac{dy}{dt} = -2(y^3 - y)$

(B) $\frac{dy}{dt} = y^3 - y$

(C) $\frac{dy}{dt} = y(1 - y)^2$

(D) $\frac{dy}{dt} = y^4 - y^2$

(E) None of the above

Quiz 1

- Please hand in Quiz 1

Quiz 1

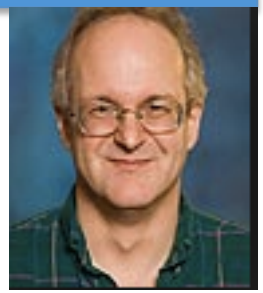
- Group version:

- Please do use laptops and spreadsheets, notes, books, etc.

XPP code
(for the aphid-ladybug computations):

```
#x=aphids
#y=ladybugs
dx/dt=r*x-y*b*x^2/(k^2+x^2)
dy/dt=y*c*x^2/(k^2+x^2)-d*y

init x=50,y=1

param r=0.5,k=20,b=30,c=5,d=2
@ xplot=t,yplot=y,zplot=x,axes=3d
@ xmin=0,xmax=100,ymin=0,ymax=5,zmin=0,zmax=60
@ xlo=-2,ylo=-2,xhi=2,yhi=2
@ phi=60
done
```

XPP is free software written by my friend Bard Ermentrout. See me for more information