The geometry of change

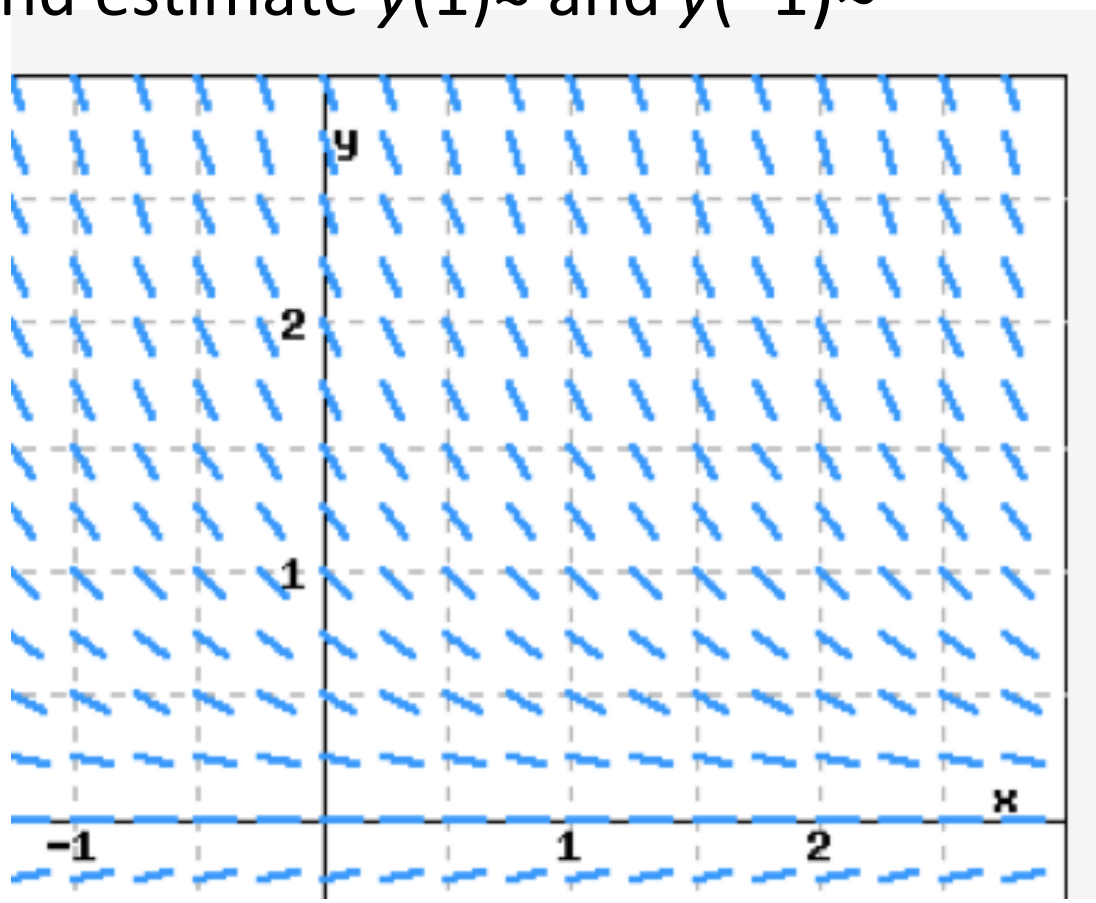
Slope fields and other diagrams for qualitative solutions to differential equations

UBC Math 102

Help with Homework

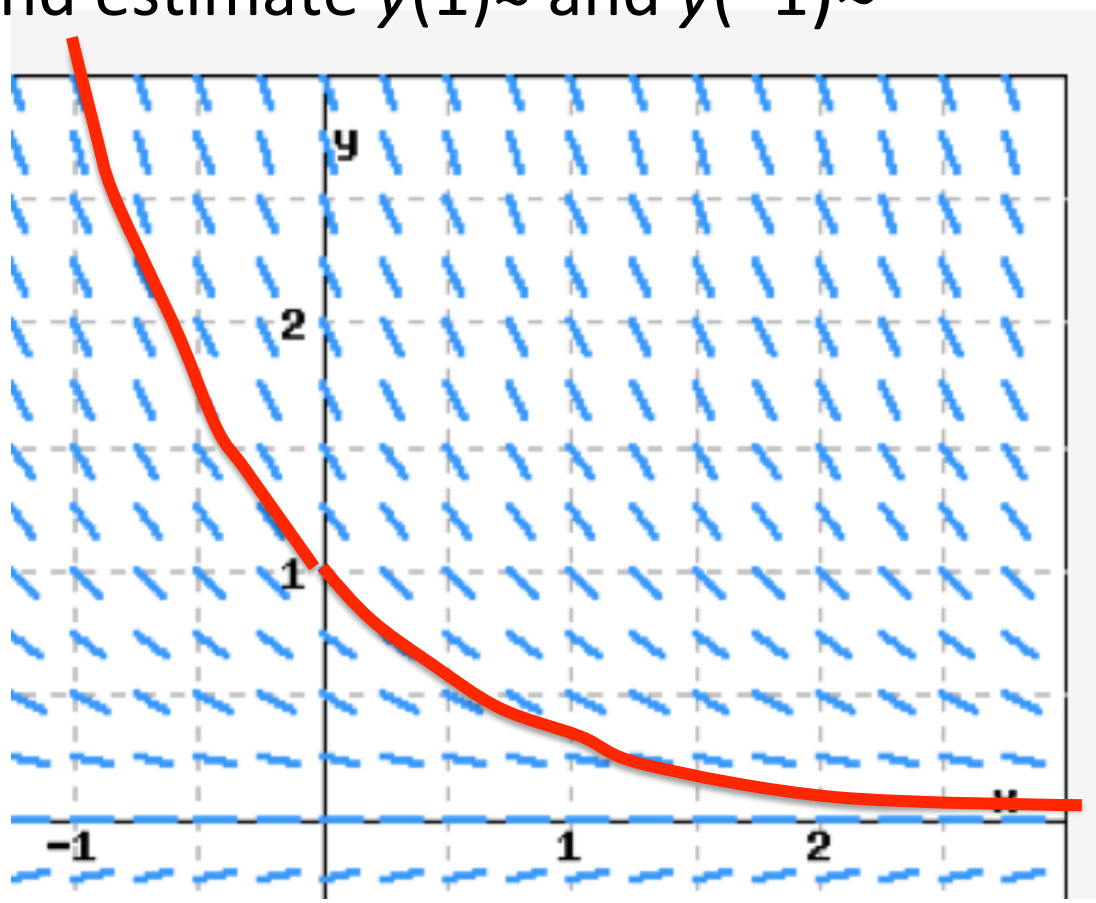
Assignment10:

For the solution that satisfies $y(0)=1$, sketch the solution curve and estimate $y(1)\approx$ and $y(-1)\approx$



Assignment10: Problem 8

For the solution that satisfies $y(0)=1$, sketch the solution curve and estimate $y(1)\approx$ and $y(-1)\approx$



The “math buddy” system

- Matchin up a few students who are comfrortable with math with those who need some help.
- Let me know if you want to be a math buddy or you want to be matched up.
- Idea: someone to consult with quick questions when you are stuck.

More problems like on the exam

- Math 102 wiki

General information

Course policies
Assignments and quizzes
Midterm information
Final exam information
Success strategies
Getting help

Practice for the final exam

MATH102/Question Challenge wiki.

Math Exam Resources

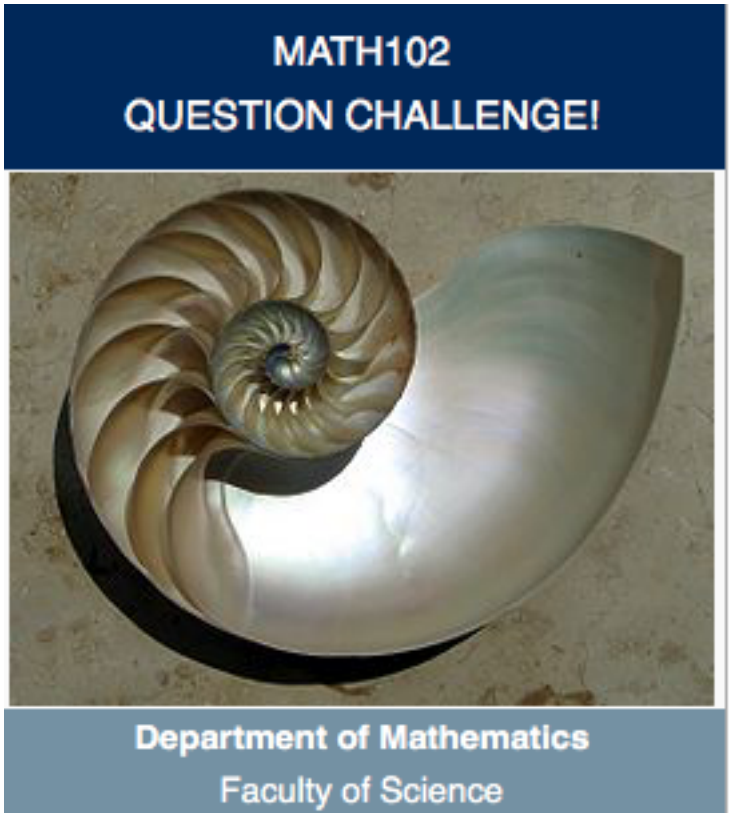
MATH102/Question Challenge wiki

More problems like on the exam

- Our unique place to work and study online:
- http://wiki.ubc.ca/Course:MATH102/Question_Challenge

Questions in the Challenge Pool

- 2001 December Q03
- 2001 December Q5
- 2002 December Q6
- 2004 December Q6
- 2008 December Q2a
- 2008 December Q2b



Math Exam Resources

The Math Exam Resource (MER)

- http://wiki.ubc.ca/Science:Math_Exam_Resources/Courses/MATH102

- Math 102 old exams: Hints and full solutions

Past Final Exams

- December 2014
- December 2013
- December 2012
- December 2011

Last time:

Differential equations and population growth

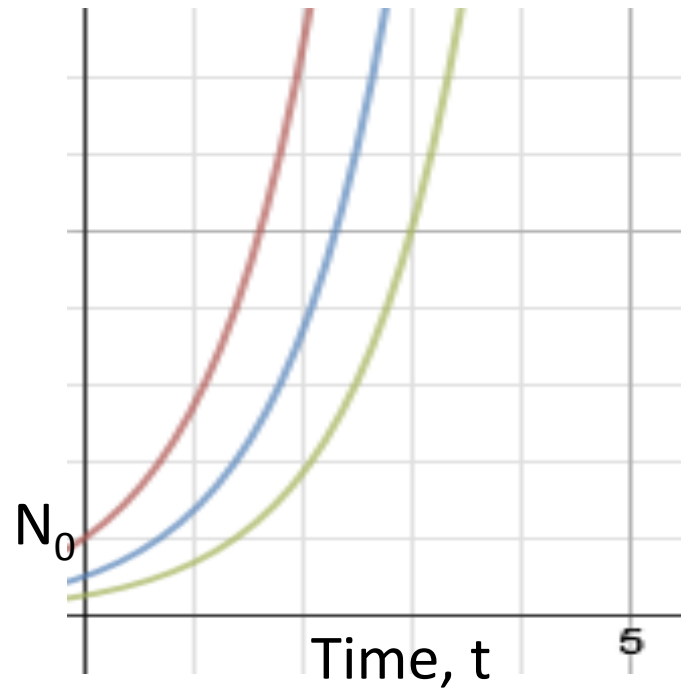
Unlimited population growth

So far we have one model for population growth. DE: $dN/dt = rN$ IC: $N(0)= N_0$

Then $N(t)=N_0 e^{rt}$

Population grows

EXPONENTIALLY



Unlimited population growth

So far we have one model for population growth. DE: $dN/dt = rN$ IC: $N(0)= N_0$

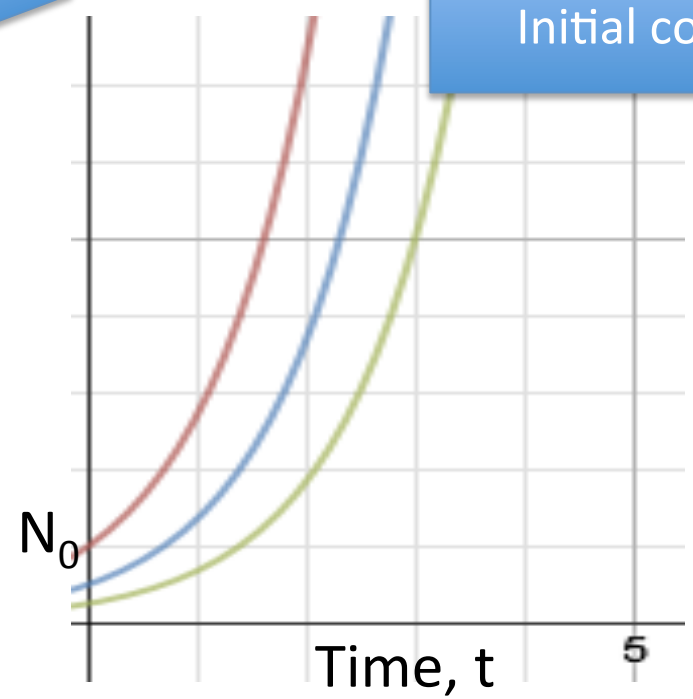
r= net Growth rate

Initial condition

Then $N(t)=N_0 e^{rt}$

Population grows

EXPONENTIALLY

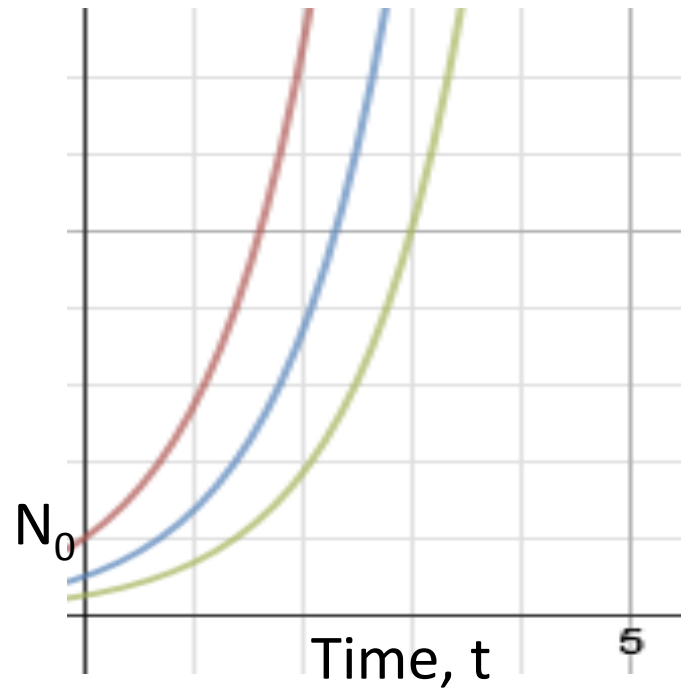


Unlimited population growth

$$dN/dt = rN \quad N(0)= N_0 \quad \rightarrow \quad N(t)=N_0\, e^{rt}$$

Critique:

1. Unrealistic growth
2. Resources limited
3. Not sustainable



Checking

- Given a function, we can check if it satisfied a DE
- So we can check if it is a solution.
- Example: the DE $\frac{dh}{dt} = -k\sqrt{h}$ represents the height of water draining out of a cylindrical container.
- The function $\boxed{h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2}$ is a solution to that DE with $h(0)=h_0$.
- We can check that this is true

Finding

- But how would we find that function if we did not know it???

- Example: the DE $\frac{dh}{dt} = -k\sqrt{h}$ represents the height of water draining out of a cylindrical container.
- The function $\boxed{h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2}$ is a solution to that DE with $h(0)=h_0$.
- We can check that this is true

The basic idea

- It is hard to find "formula" solutions to MOST differential equations.
- Sometimes we just want to know the overall (qualitative) behaviour..
- We can deduce some of this from properties of the differential equation itself? Without finding the full solution?

Example: Immigration in Europe

In Europe, birth rates are lower than mortality, but there is a constant rate of immigration, I

$$dN/dt = I - \mu N \quad \text{where } \mu = m - r > 0$$

$$I > 0$$

Constant rate of immigration

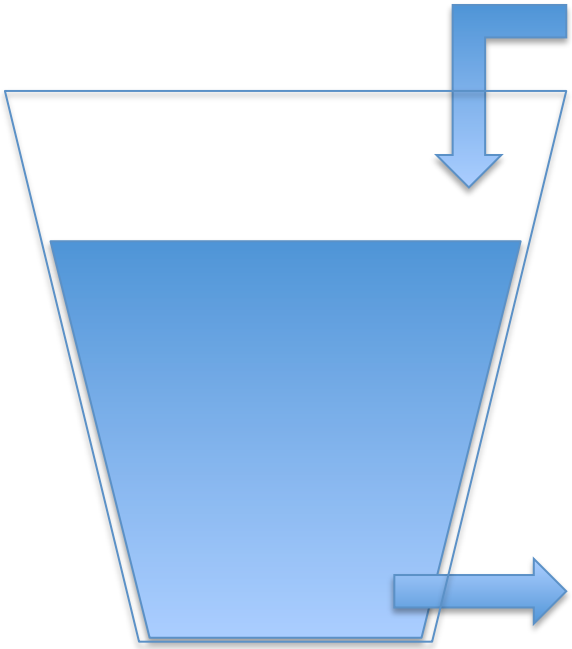
Constant rate of mortality

What it says:

Rate of change
Of population = rate immigr – rate mortality

$$dN/dt = I - \mu N$$

rate immigration



rate of mortality

Recall

$$dN/dt = I - \mu N$$

- This is a differential equation (DE). It keeps track of (instantaneous) changes
- We are interested in fining a function of time N(t) that satisfies this equation and predicts the behaviour of the population over time.
- We call such a function a solution of the differential equation.

Special solutions

- In general, it is not easy to find the solution to a DE.
- But some solutions are “easy to find”

- For example: a solution in which thre is NO CHANGE over time (where $dN/dt = 0$).

Example: from last time

For what population level is there no change?

$$dN/dt = I - \mu N$$

$$dN/dt = 0$$

No change

There is no change when immigration and mortality exactly balance

$$0 = dN/dt = I - \mu N$$

$$I = \mu N$$

$$N = I/\mu$$

There is no change when immigration and mortality exactly ballance

We say that

$$N = I / \mu$$

is a steady state

$$dN/dt = I - \mu N$$

Steady state solution

- Population vs time:

What if immigration and mortality do not balance?

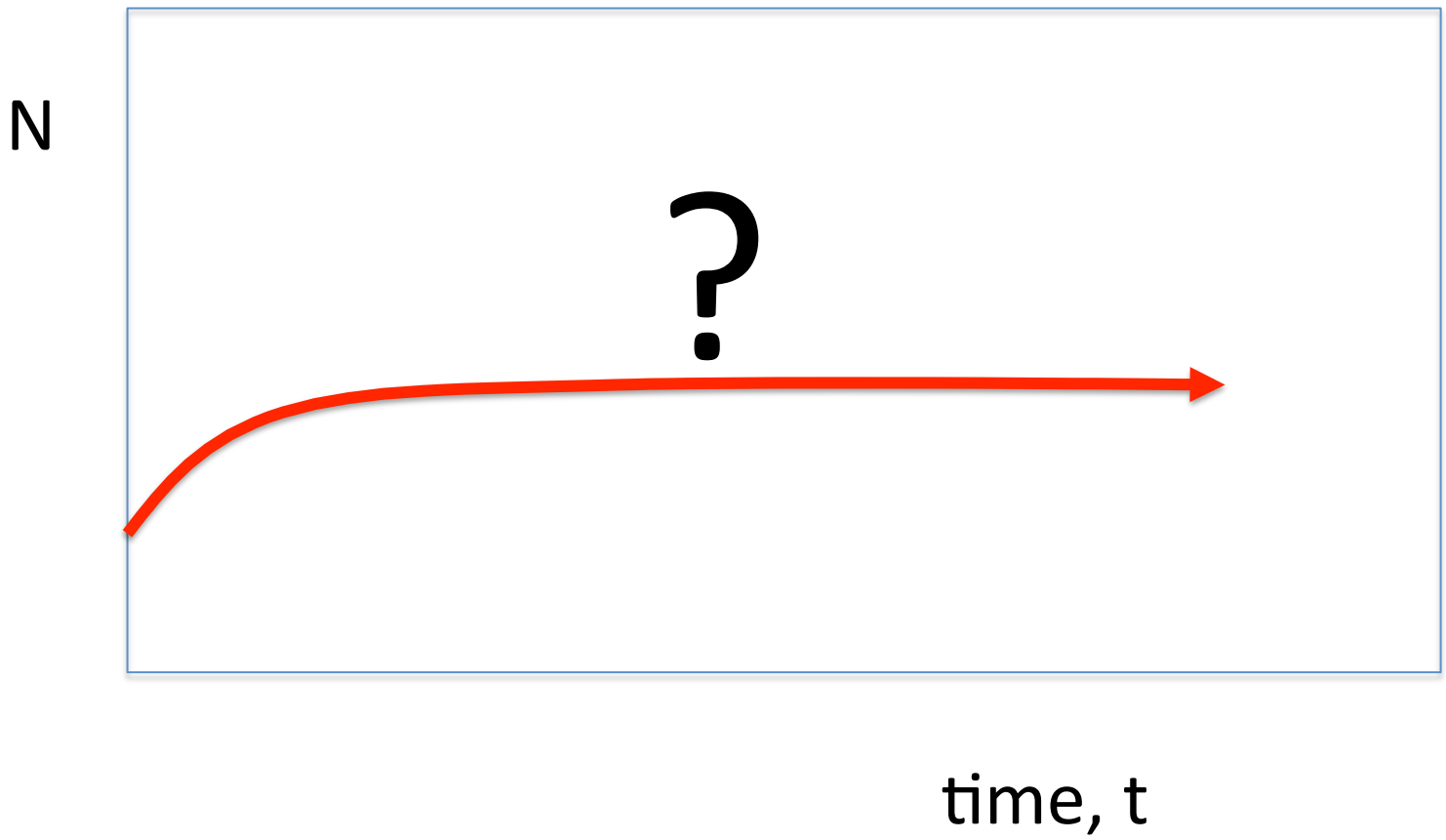
- How do we figure out what happens to the population over time?

- (intuitively: we expect N to either increase or decrease!)

$$dN/dt = I - \mu N$$

Want to find a function N(t)

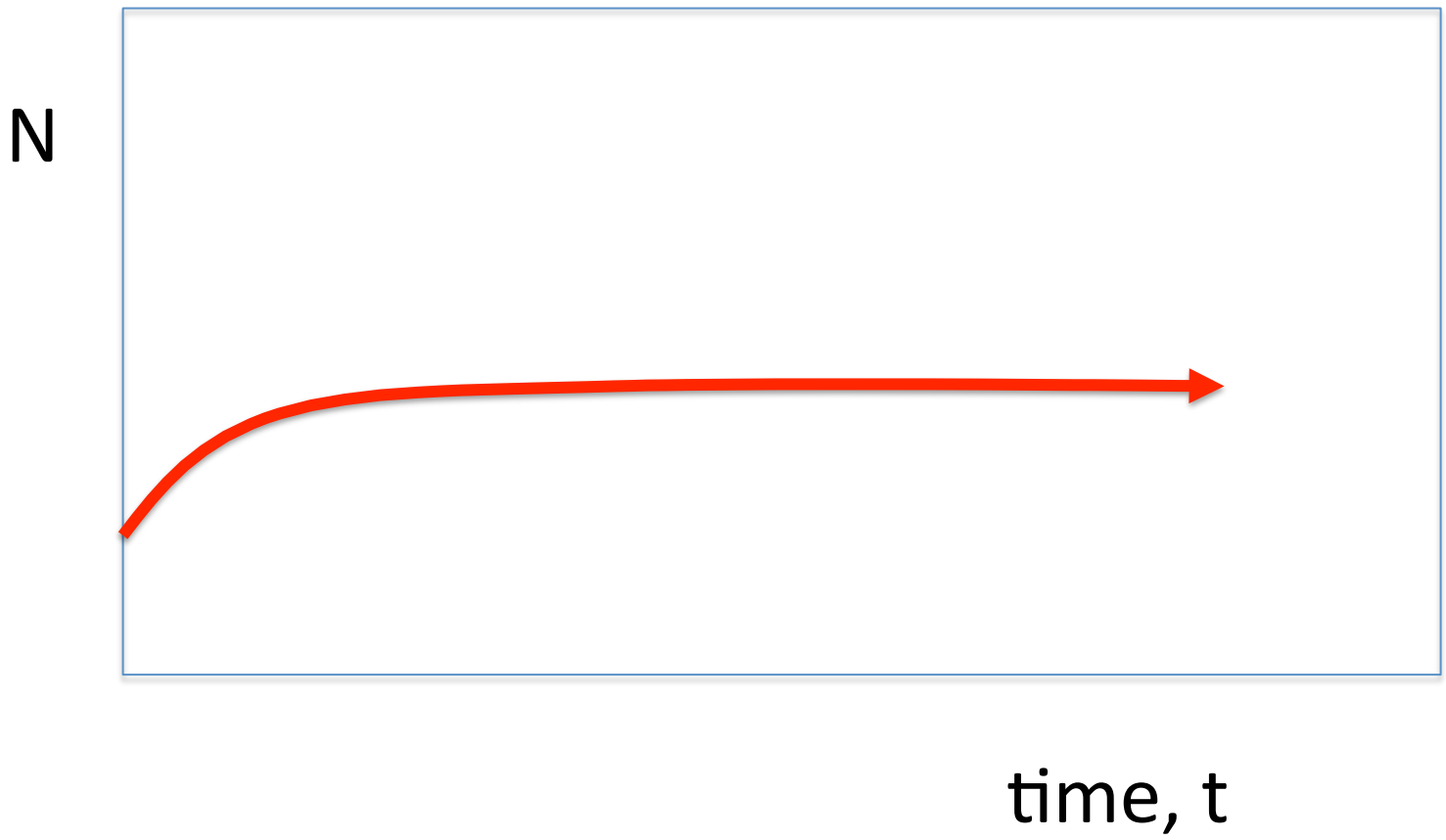
- That describes the behaviour of the population



$$dN/dt = I - \mu N$$

Want to find a function N(t)

- We might be able to find that function, but we can easily SKETCH ITS GRAPH!!



We will do so by thinking of slopes

Then DE describes slopes of N vs t

$$dN/dt = I - \mu N$$

$N > I/\mu$ $\quad dN/dt > 0$ $\quad \nearrow$

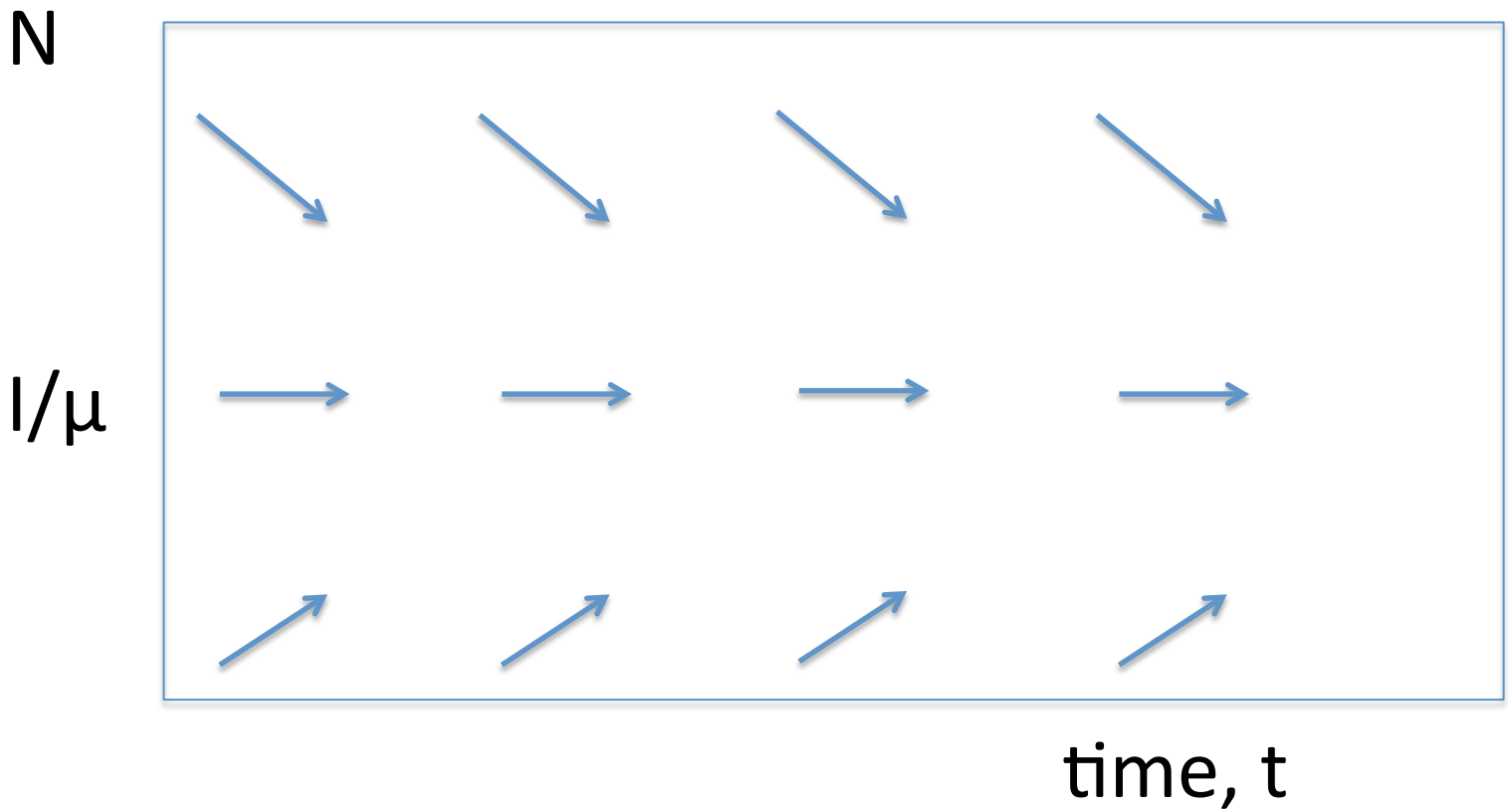
$N = I/\mu$ $\quad dN/dt = 0$ $\quad \rightarrow$

$N < I/\mu$ $\quad dN/dt < 0$ $\quad \searrow$

This slide has been corrected from before

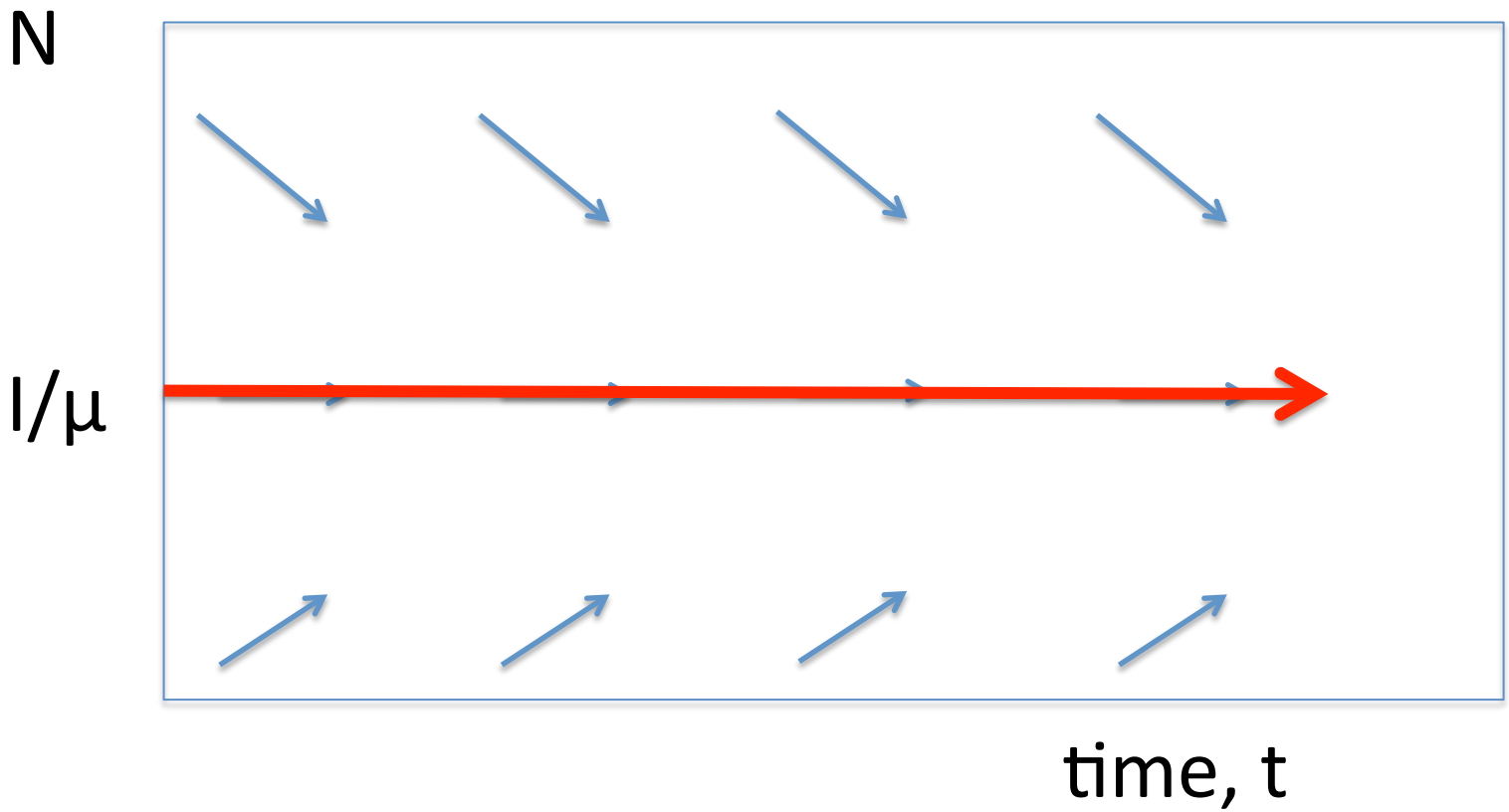
Put this on a graph N vs t

- Inidicate the slopes on a graph of N vs t



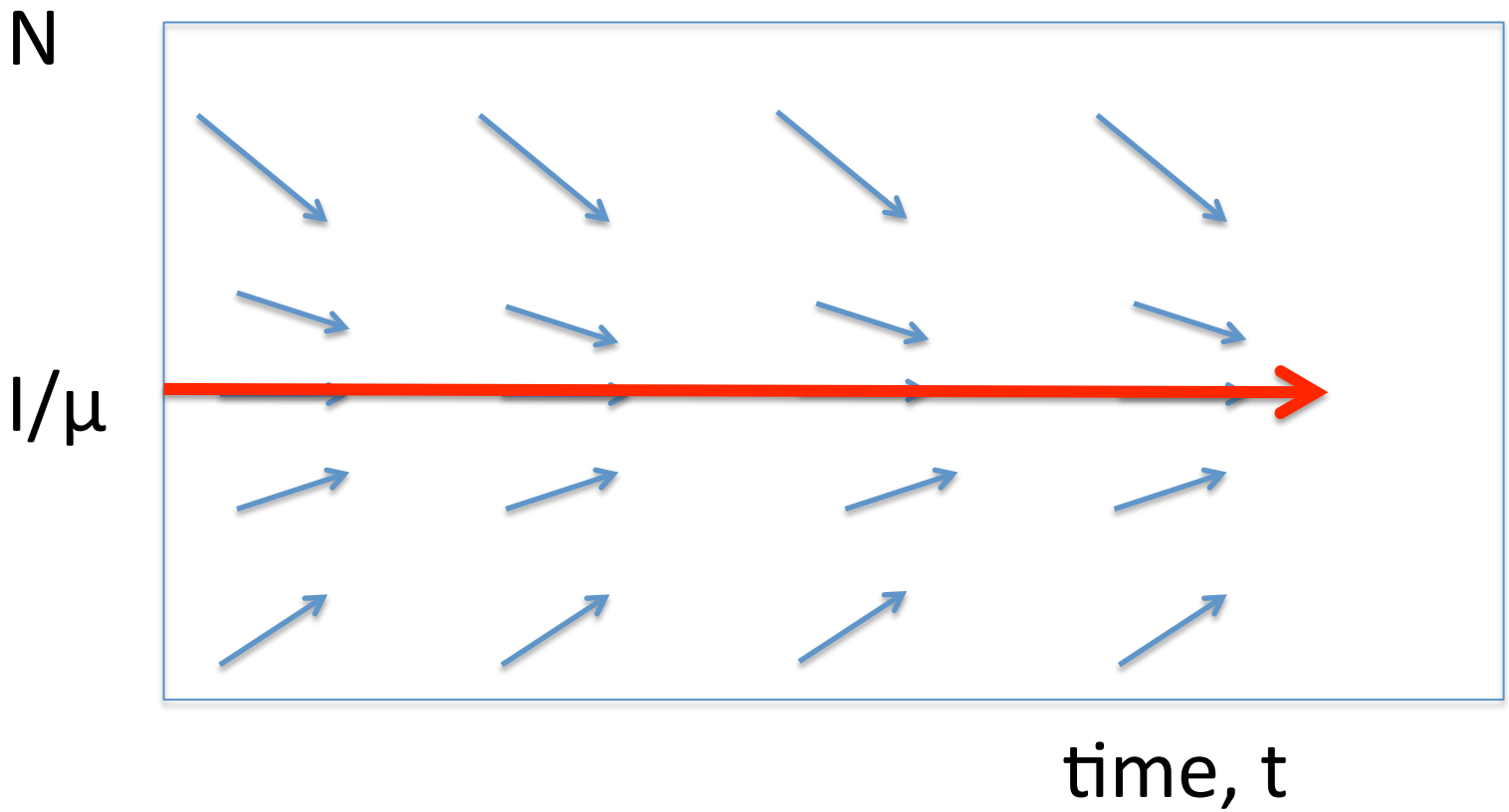
If we start at the steady state, there is no change

- $N(0)=I/\mu$



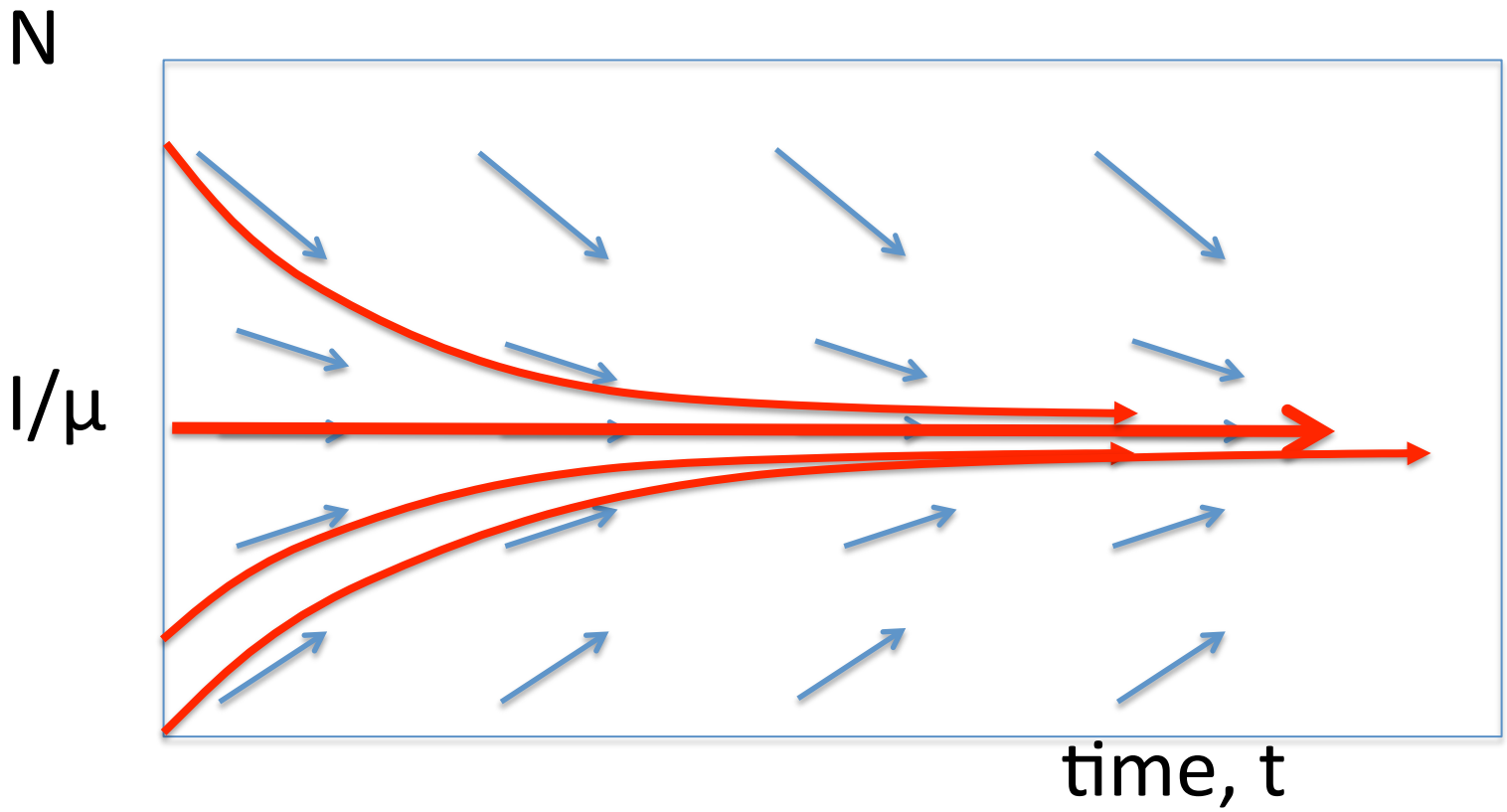
We could add more slopes

- This diagram is called a "slope field"



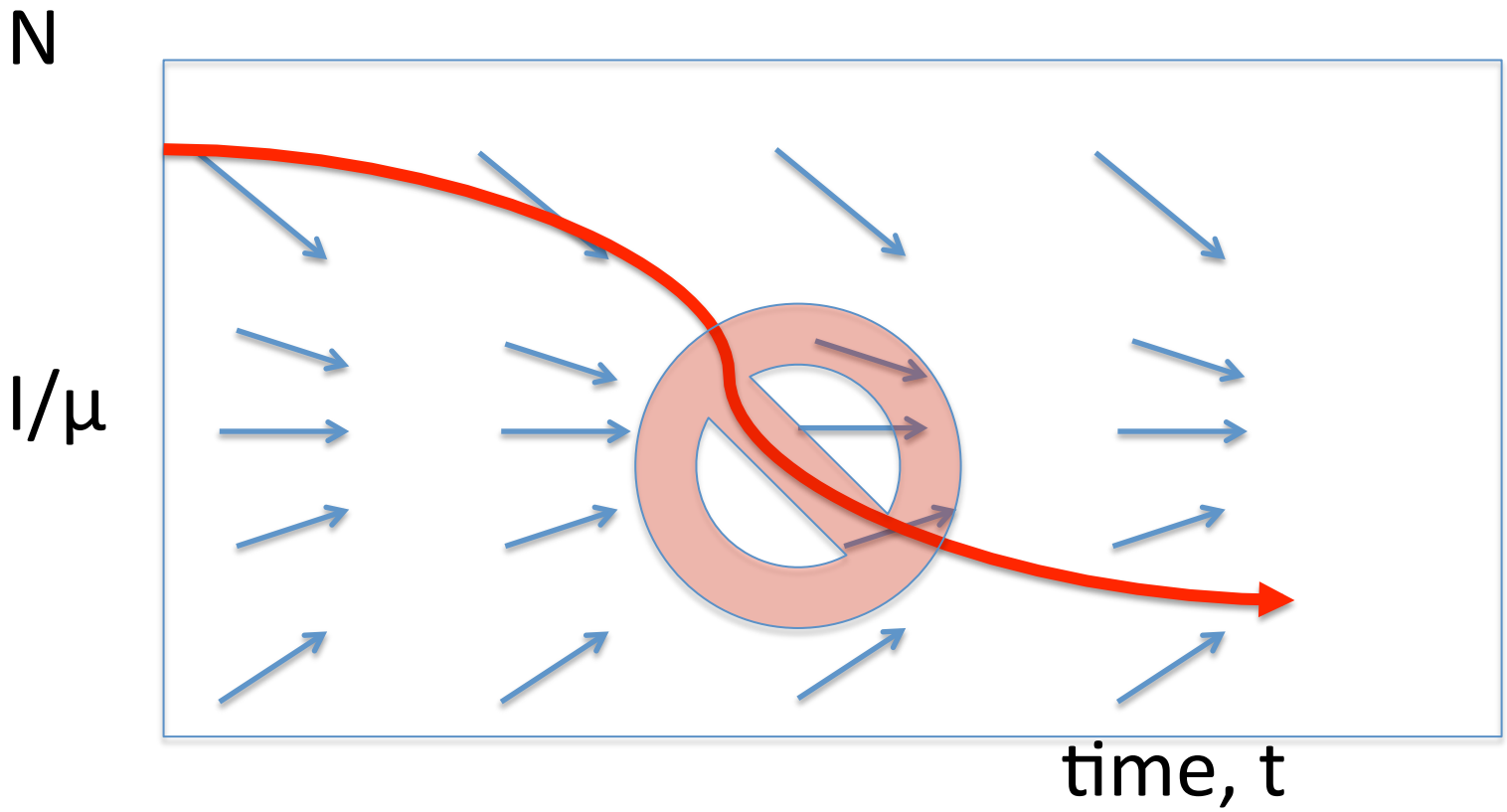
Slope field=tangent vectors

Solutions N(t) has to keep these as tangents



Slope field=tangent vectors

Not allowed to “cross” anoy of those tangents



The logistic equation

A model for density-dependent growth

UBC Math 102

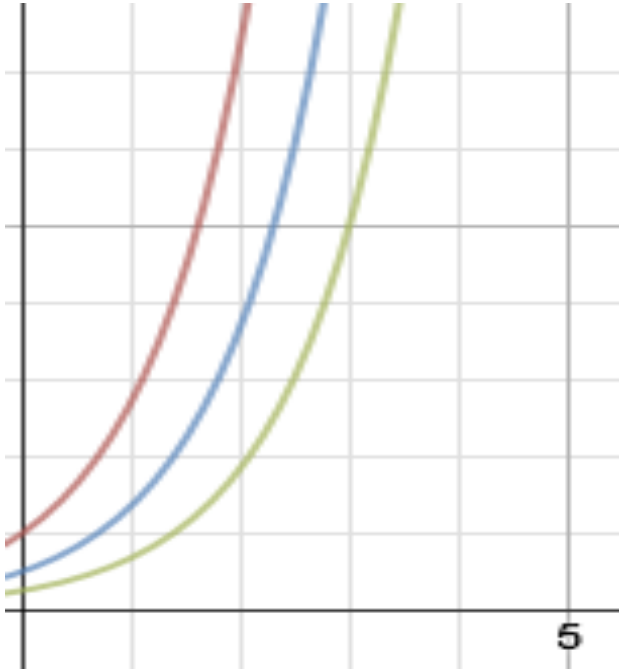
Population growth

So far we have one model for population growth.

$$dN/dt = rN$$

Critique:

1. Unrealistic growth
2. Resources limited
3. Not sustainable



Revised “density dependent” growth

Hypotheses:

- Environment has limited resources to sustain population
- Growth cannot continue at constant per capita rate indefinitely
- Beyond some population size, there will be net mortality

Logistic growth

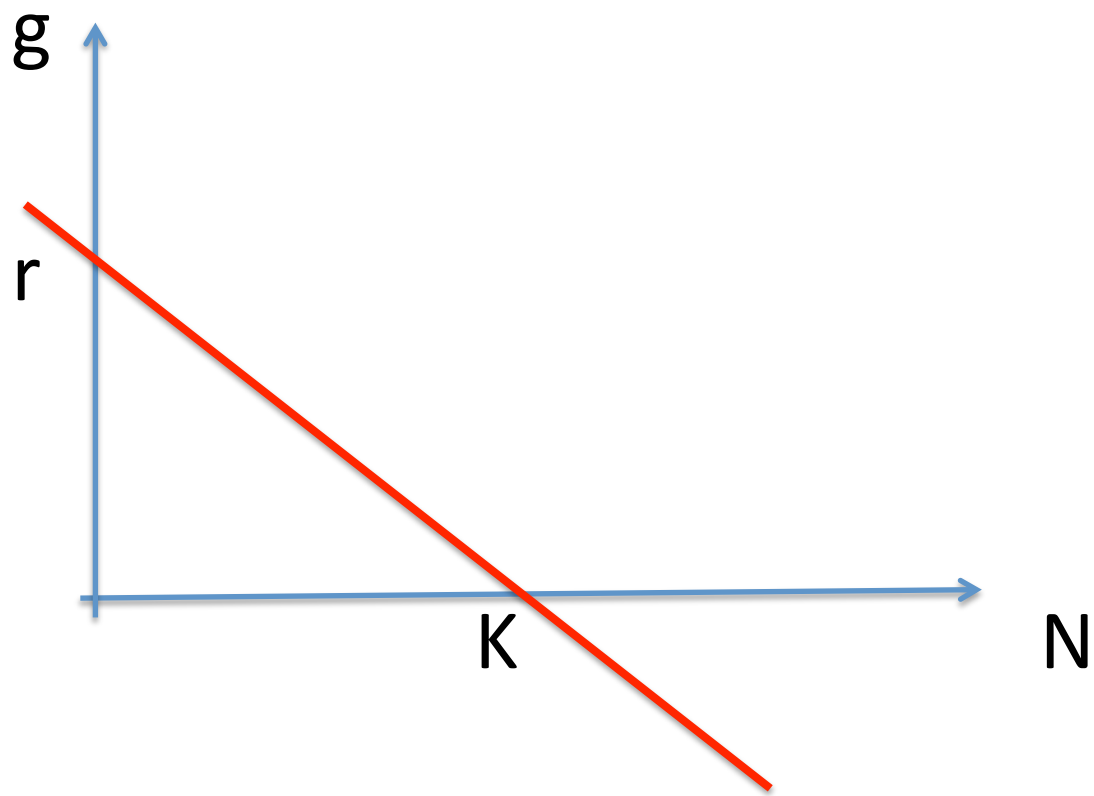
Similar idea: $dN/dt = gN$

BUT now assume that *growth rate* ***g*** is not constant!

Assume that the per-capita growth rate decreases as population size increases.

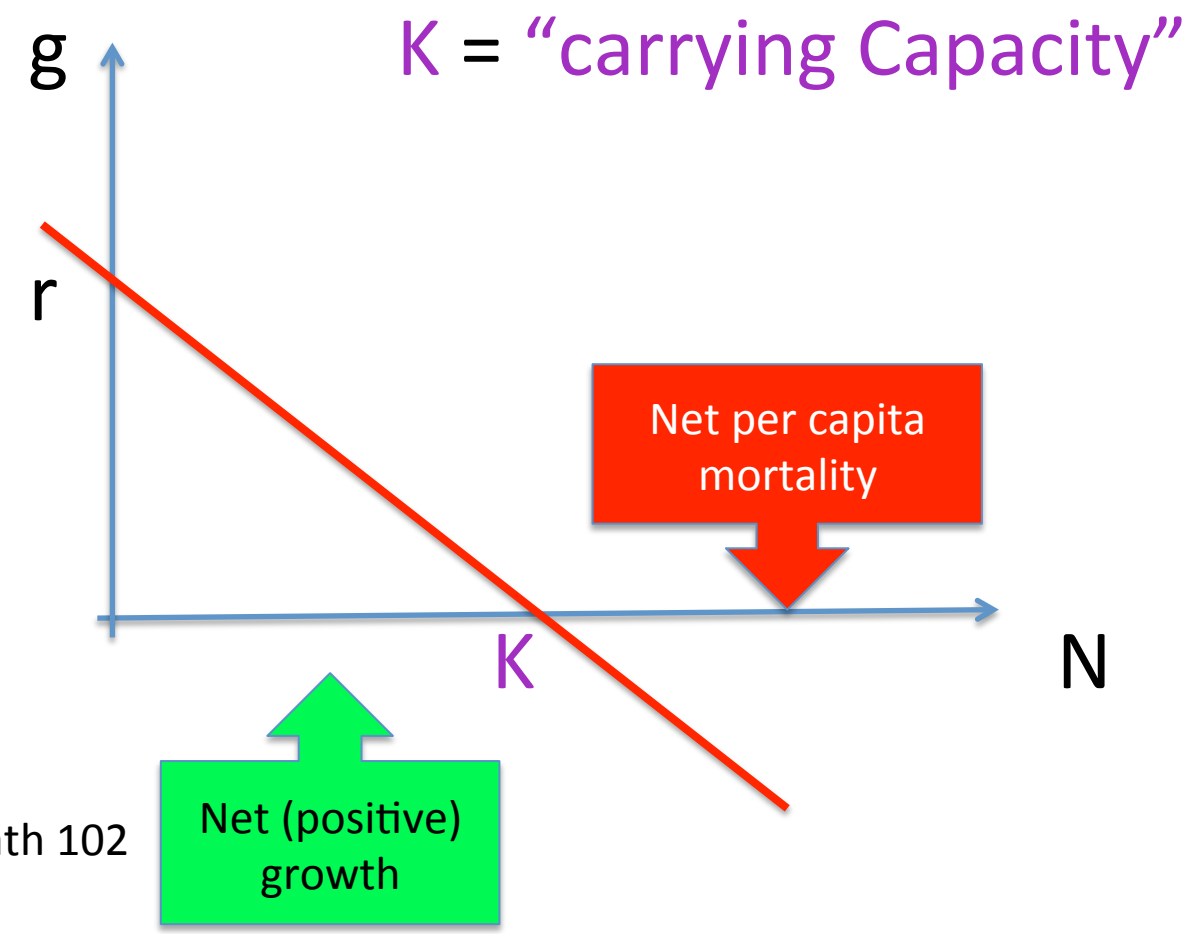
“Density dependent” growth rate

- Assume g(N) decreases linearly with N:



"Density dependent" per capita growth rate

- Assume g(N) decreases linearly with N:



(4) The function g(N) shown here is:

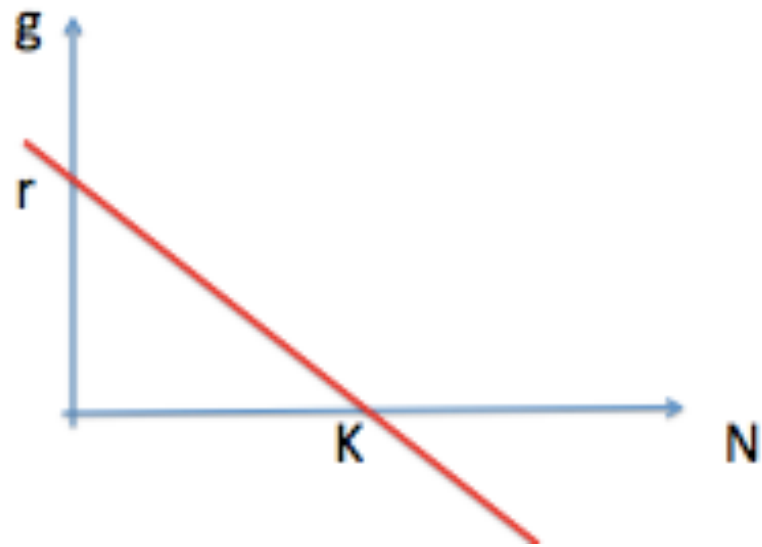
(A) $g(N) = rN-K$

(B) $g(N) = KN-r$

(C) $g(N)=(r-K)N$

(D) $g(N)=r(N-K)$

(E) $g(N)=r(K-N)/K$



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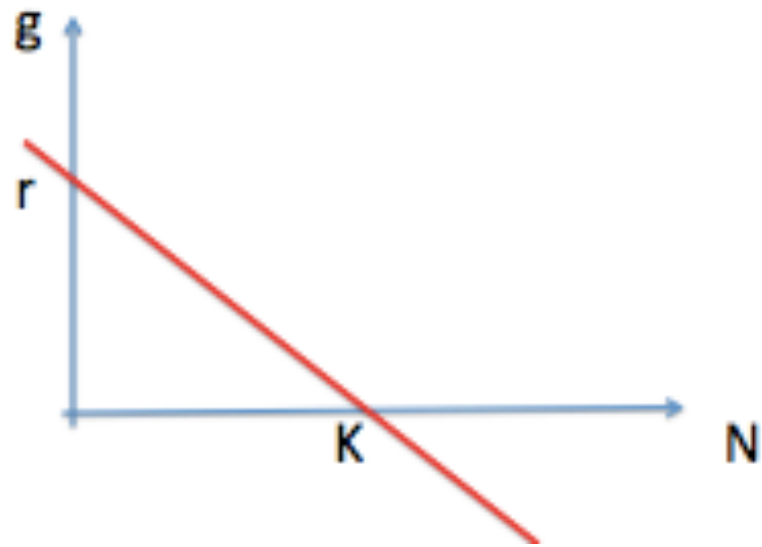
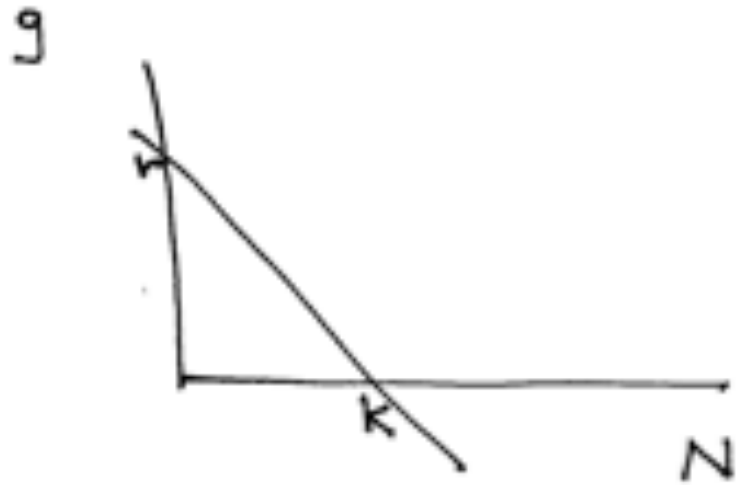


Solution:



$$g = \underbrace{r}_{\text{interc}} + \underbrace{-\frac{r}{K}}_{\text{slope}} N$$

$$= r\left(1 - \frac{N}{K}\right)$$

$$g(N) = r\left(\frac{K - N}{K}\right)$$

(5) So the new diff'l eqn we want is:

(A) $dN/dt=rN(K-N)/K$

(B) $dN/dt=r(N-K)/K$

(C) $dN/dt=rN(N-K)$

(D) $dN/dt=rN$

(E) Not sure

(5) So the new diff'l eqn we want is:

(A) $dN/dt=rN(K-N)/K$

(B) $dN/dt=r(N-K)/K$

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So the new diff'l eqn we want is:

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(B) dN/dt=r(K-N)/K

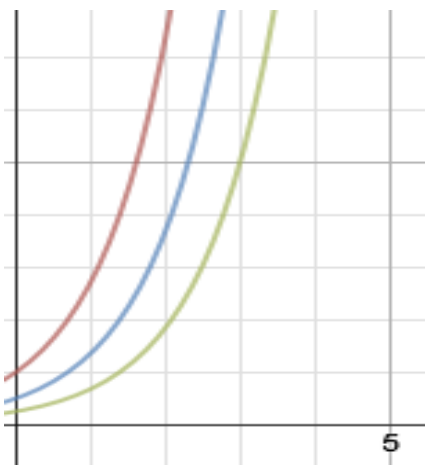
(C) dN/dt=rN(N-K)/K

(D) dN/dt=rN

(E) Not sure

$$\boxed{\frac{dN}{dt} = \left[r\frac{(K-N)}{K}\right]N.}$$

Population growth

Unlimited growth	Density dependent growth
$$\frac{dN}{dt} = rN$$	$$\frac{dN}{dt} = \left[r\frac{(K-N)}{K}\right]N.$$
	K=carrying capacity r= intrinsic growth rate

Other ways of interpreting the logistic equation:

Expand:

$$\frac{dN}{dt} = rN - \frac{r}{K}N^2$$

Simple birth term (just like before)

Mortality when overcrowded

Create a table and and draw slope field for logistic equation

Fill in this table:

$$\frac{dN}{dt} = \left[r \frac{(K - N)}{K} \right] N.$$

N	dN/dt	behaviour	tangent line
0	0	---	→
0<N<K			
N=K			
N>K			

Solution:

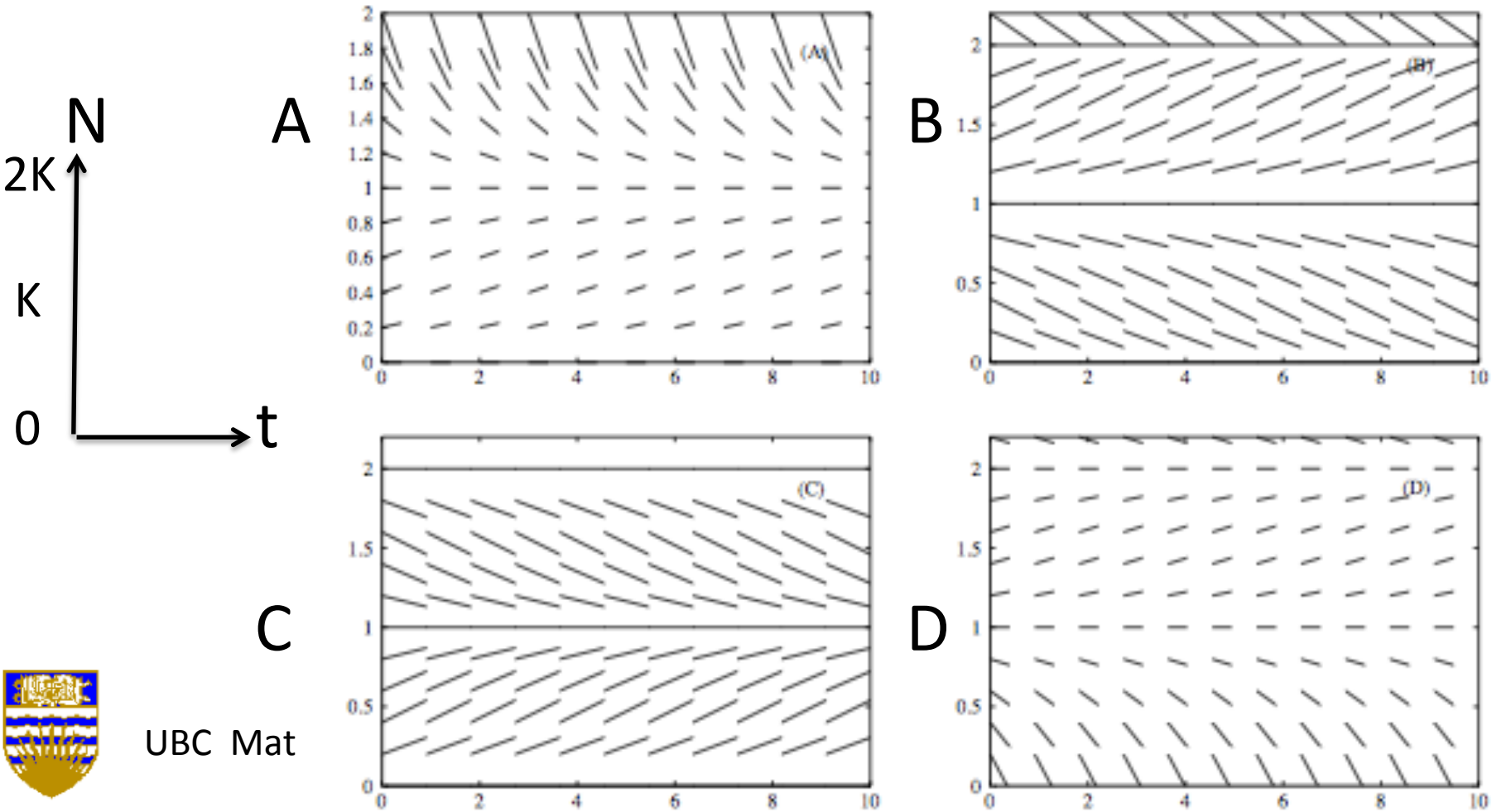
$$\frac{dN}{dt} = \left[r \frac{(K - N)}{K} \right] N.$$

N	$\frac{dN}{dt}$	beh.	tang.
0	0	—	$\rightarrow$
$0 < N < K$	$+$	incr.	$\nearrow$
K	0	—	—
$N > K$	$-ve$	decr.	$\searrow$

(6) Slope field (logistic)

$$\frac{dN}{dt} = \left[r \frac{(K-N)}{K} \right] N.$$

Which plot?



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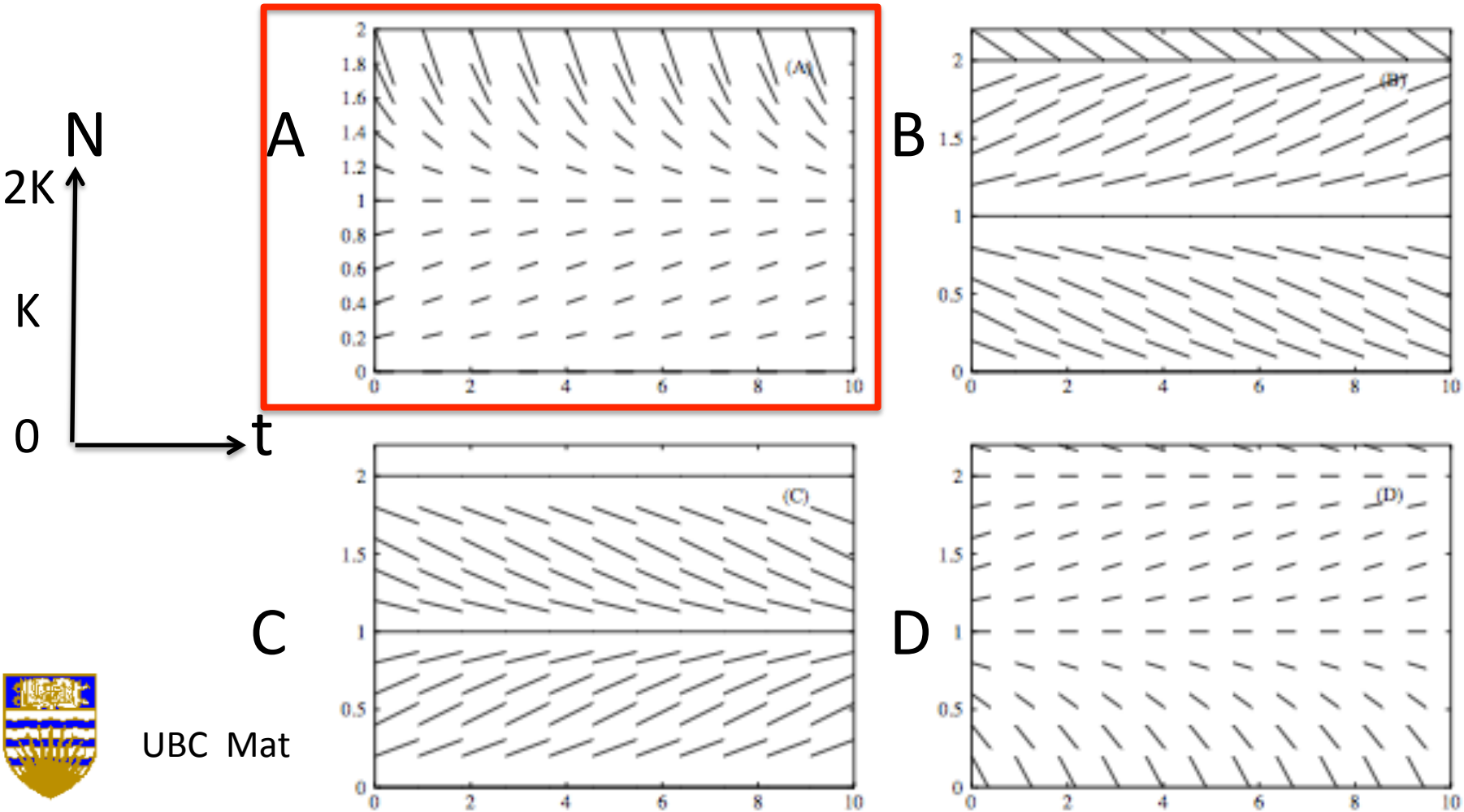
(6) Slope field (logistic)

$$\frac{dN}{dt} = \left[r \frac{(K - N)}{K} \right] N.$$

Which plot?



UBC Mat

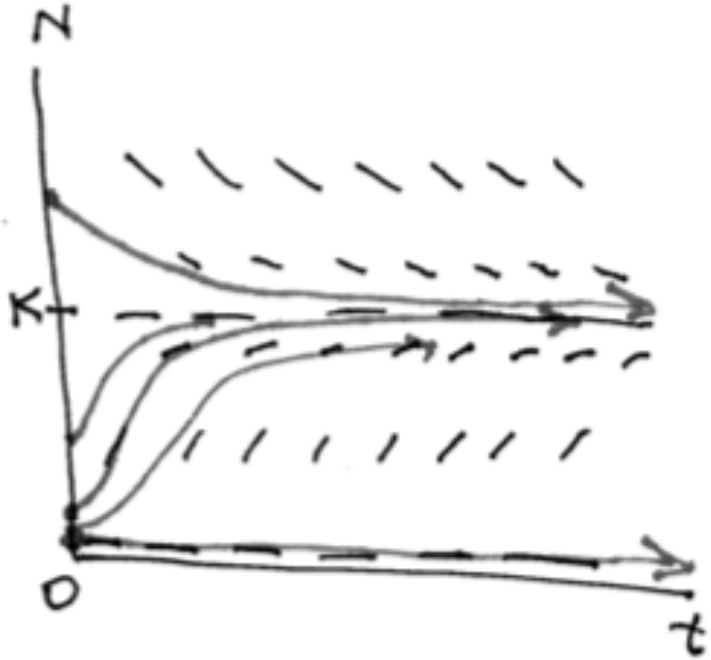
Solution curves

Use your slope field to draw a few solution curves N(t) to the logistic equation.

Then interpret the behaviour in plain English.

Explain what happens to the population after a long time.

Slope field and some solutions



Challenge: show that only solution curves that start with 0<N<K/2 have an inflection point.

Experiments

G.F. Gause *November, 1934*

"The struggle for existence"

http://www.ggause.com/Contgau.htm

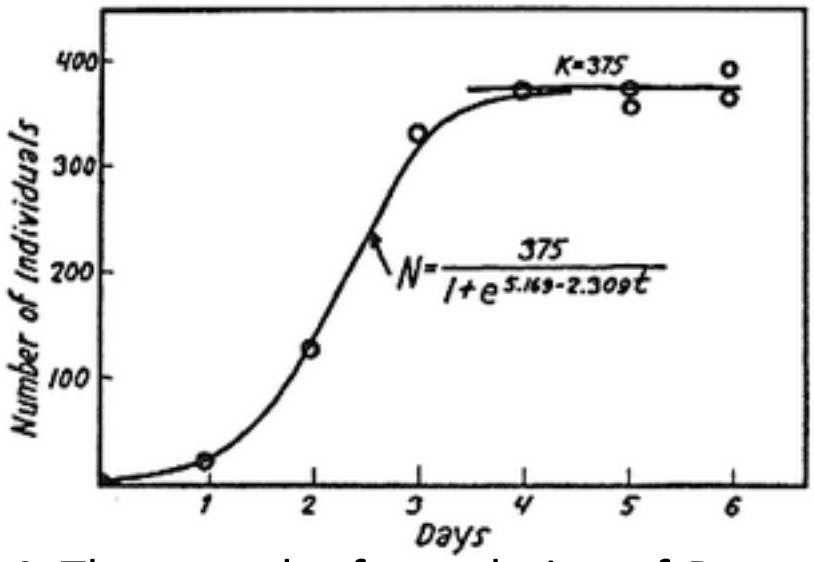


Fig. 4. The growth of population of *Paramecium caudatum*

Example:

$$\frac{dy}{dt} = f(y) = y - y^3.$$

- Use the same kind of geometric (qualitative) approach.

(Video)

$$\frac{dy}{dt} = f(y) = y - y^3.$$

y	dy/dt	behaviour	tangent
-1			
0			
0.5			
1			
2			

Tabulate values

$$\frac{dy}{dt} = f(y) = y - y^3.$$

y	sign of $f(y) = y - y^3$	behaviour of y	direction of arrow
$y < -1$	+ve	increasing	$\nearrow$
-1	0	no change in y	$\rightarrow$
-0.5	-ve	decreasing	$\searrow$
0	0	no change in y	$\rightarrow$
0.5	+ve	increasing	$\nearrow$
1	0	no change in y	$\rightarrow$
$y > 1$	-ve	decreasing	$\searrow$

$$\frac{dy}{dt} = f(y) = y - y^3.$$

How many steady states?

- (A) None
- (B) One
- (C) Two
- (D) three
- (E) Not sure

$$\frac{dy}{dt} = f(y) = y - y^3.$$

How many steady states?

- (A) None
- (B) One
- (C) Two
- (D) three
- (E) Not sure

$0 = y - y^3$

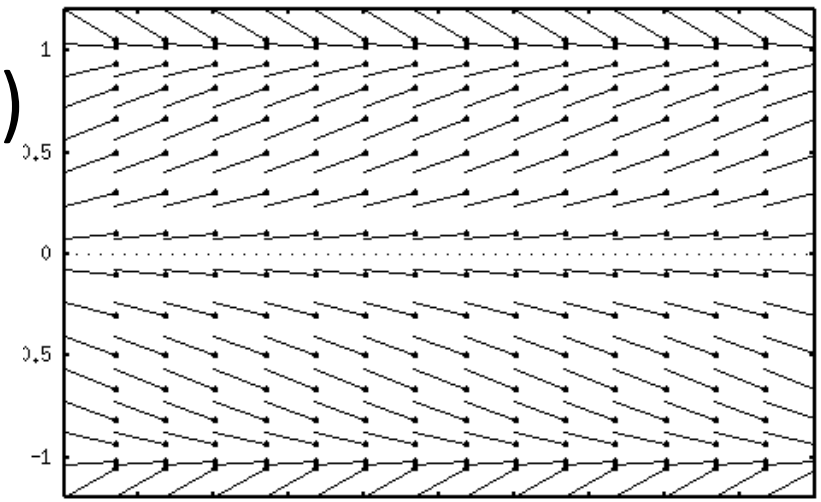
$0 = y\,(1 - y^2)$

So $y = 0, 1, -1$

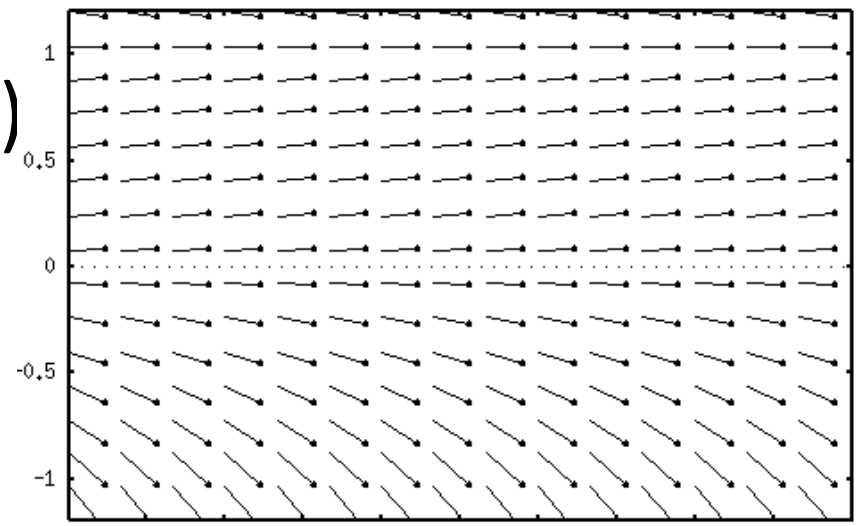
$$\frac{dy}{dt} = f(y) = y - y^3.$$

(2) The slope field should look like:

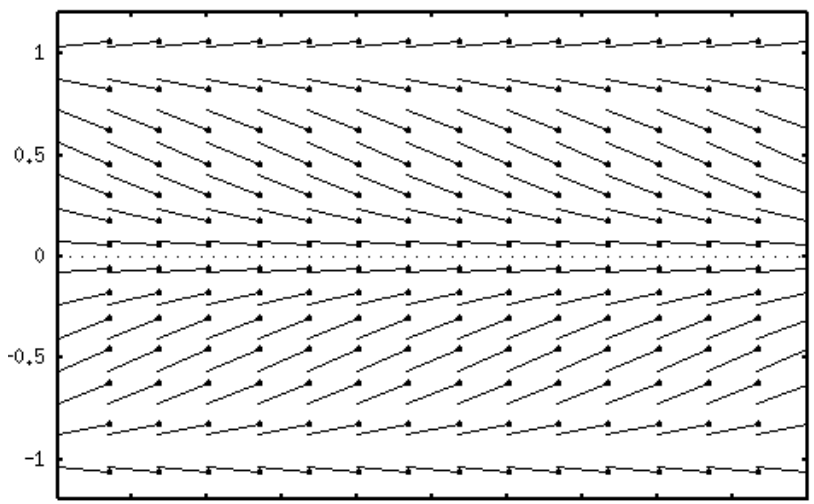
(A)



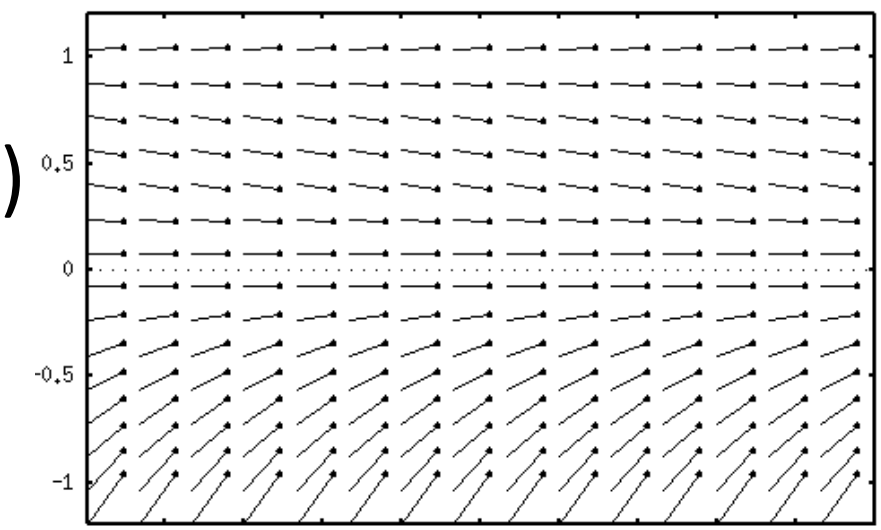
(B)



(C)



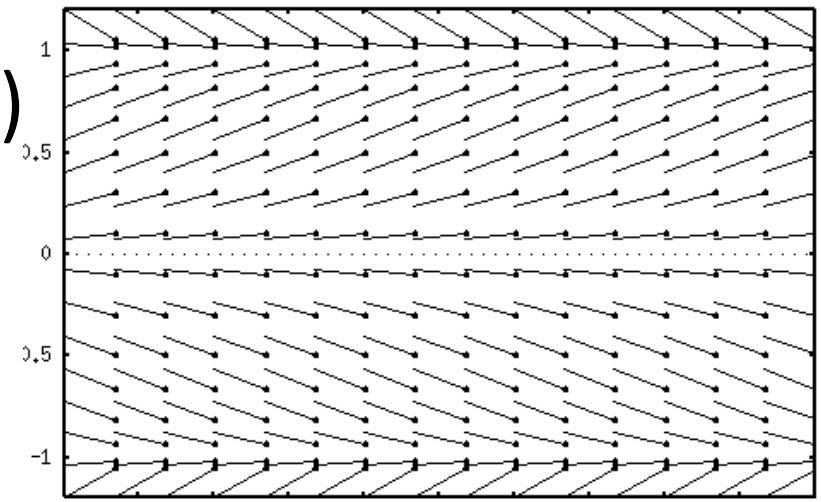
(D)



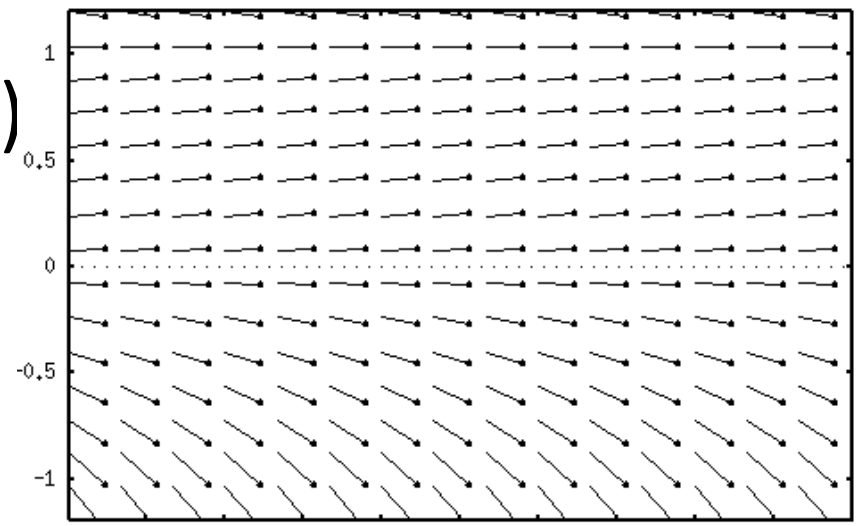
$$\frac{dy}{dt} = f(y) = y - y^3.$$

(2) The slope field should look like:

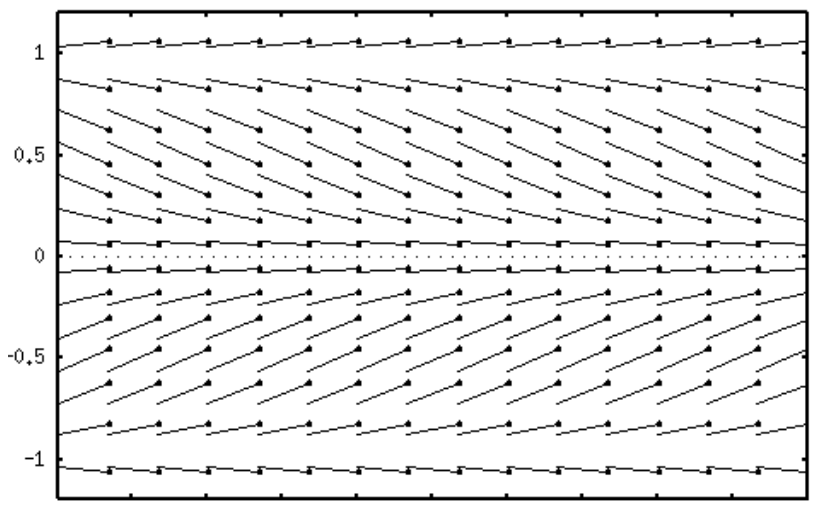
(A)



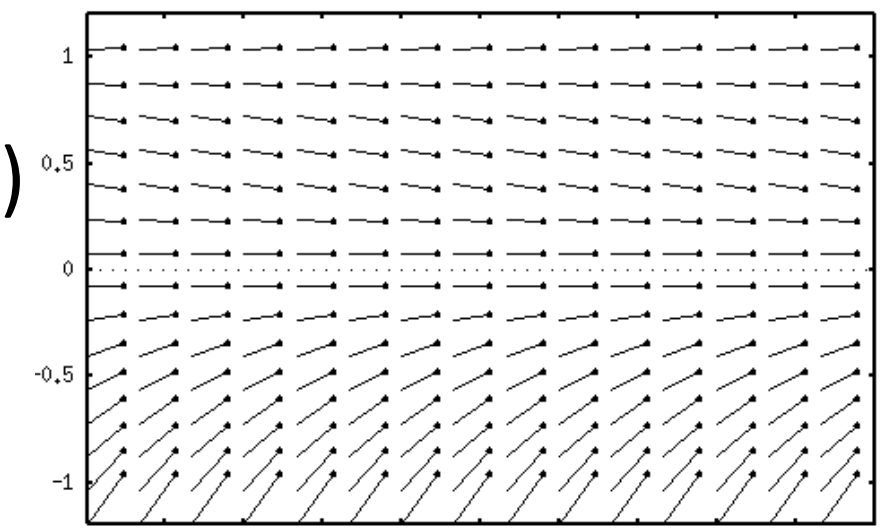
(B)



(C)

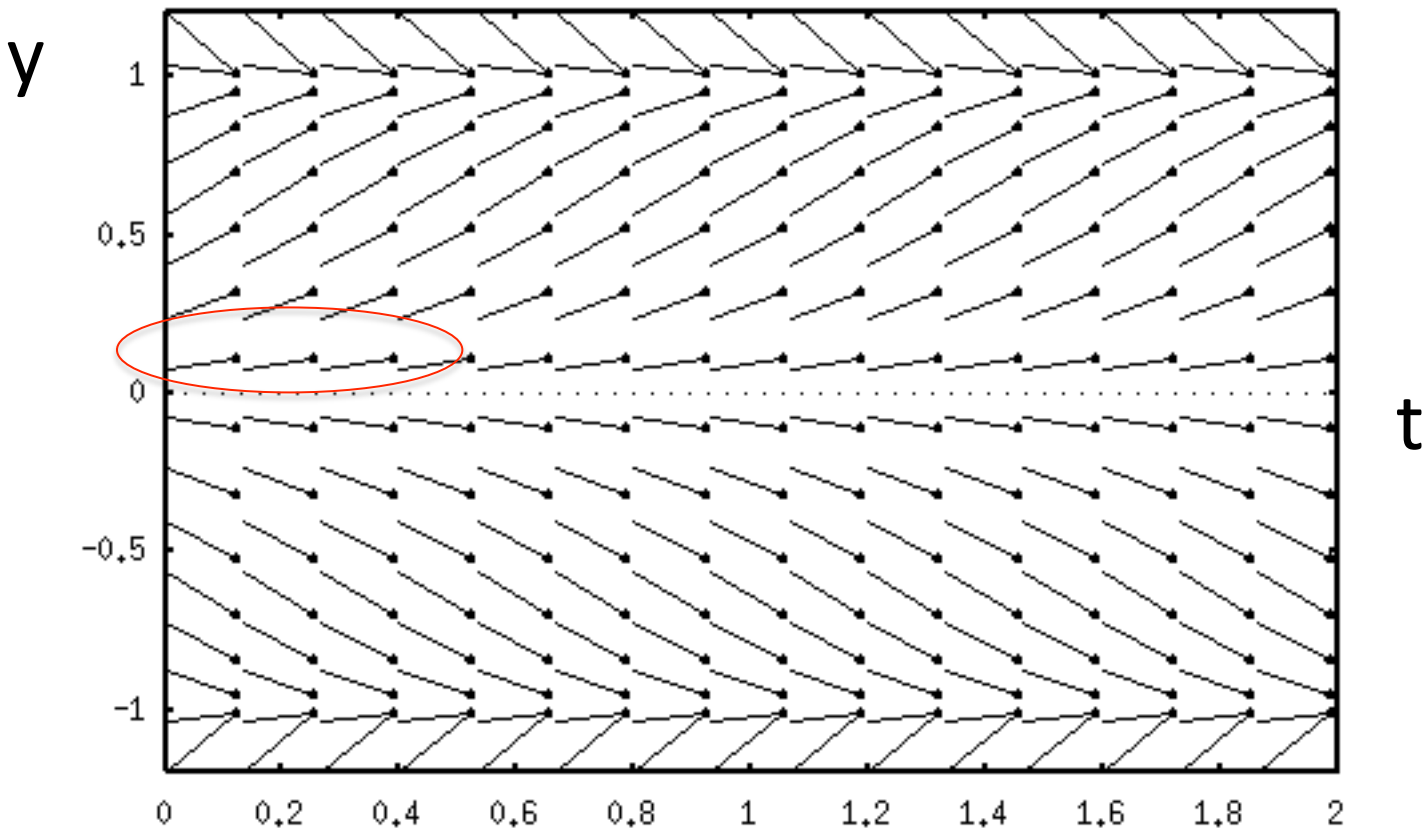


(D)



How do we know?

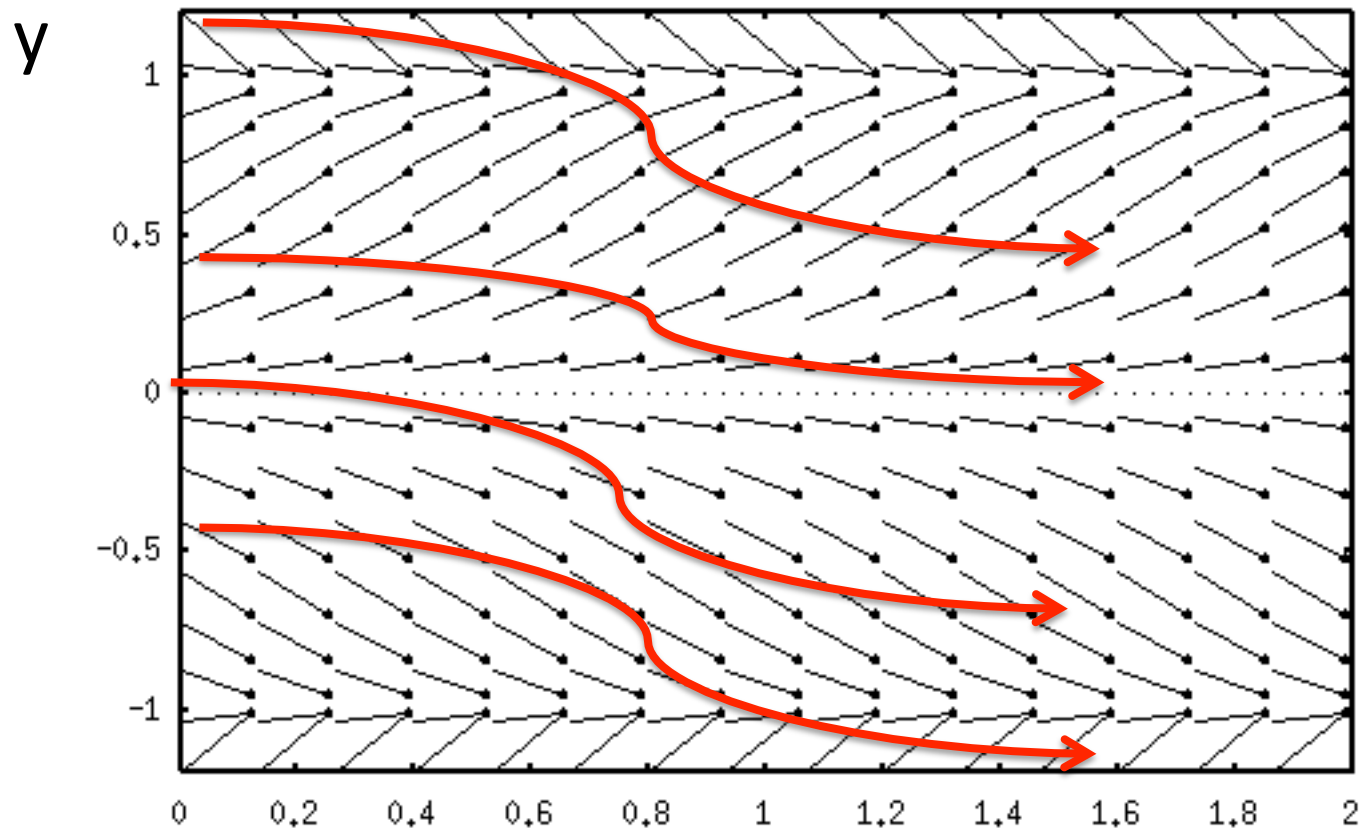
$$\frac{dy}{dt} = f(y) = y - y^3.$$



For small y, the DE is roughly $\frac{dy}{dt} = y$ so y increasing

(3) Solutions look like this?

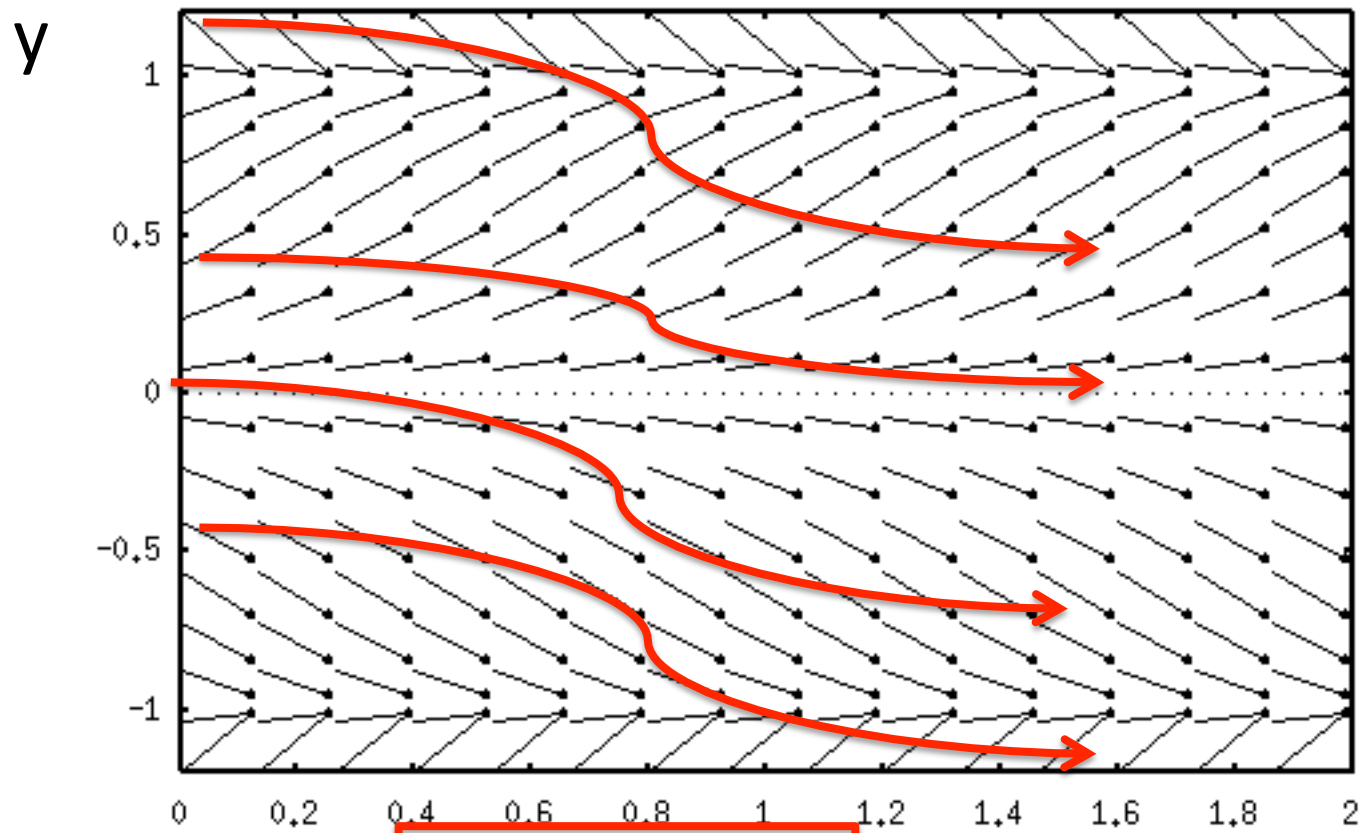
$$\frac{dy}{dt} = f(y) = y - y^3.$$



(A) TRUE (B) FALSE (C) Not sure

(3) Solutions look like this?

$$\frac{dy}{dt} = f(y) = y - y^3.$$



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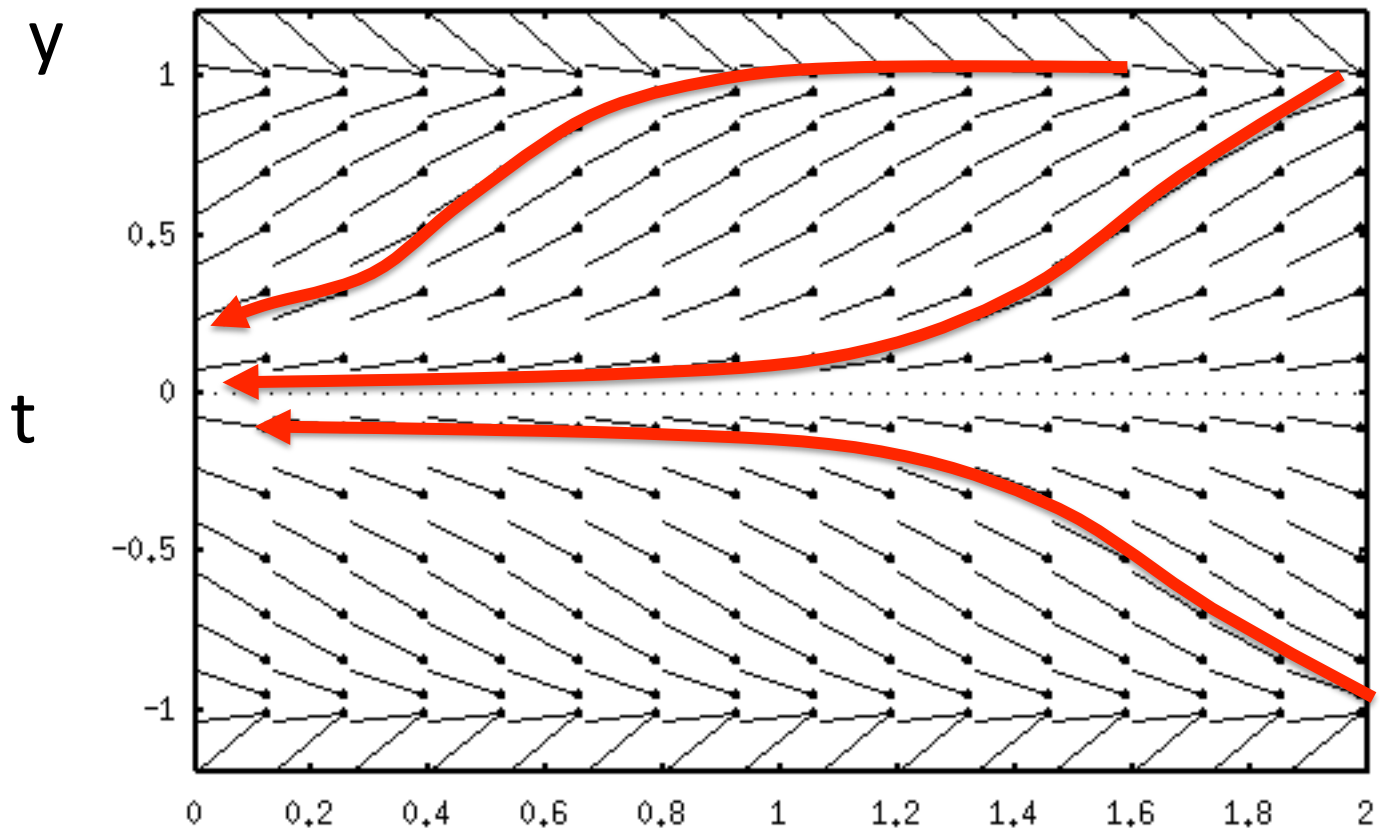

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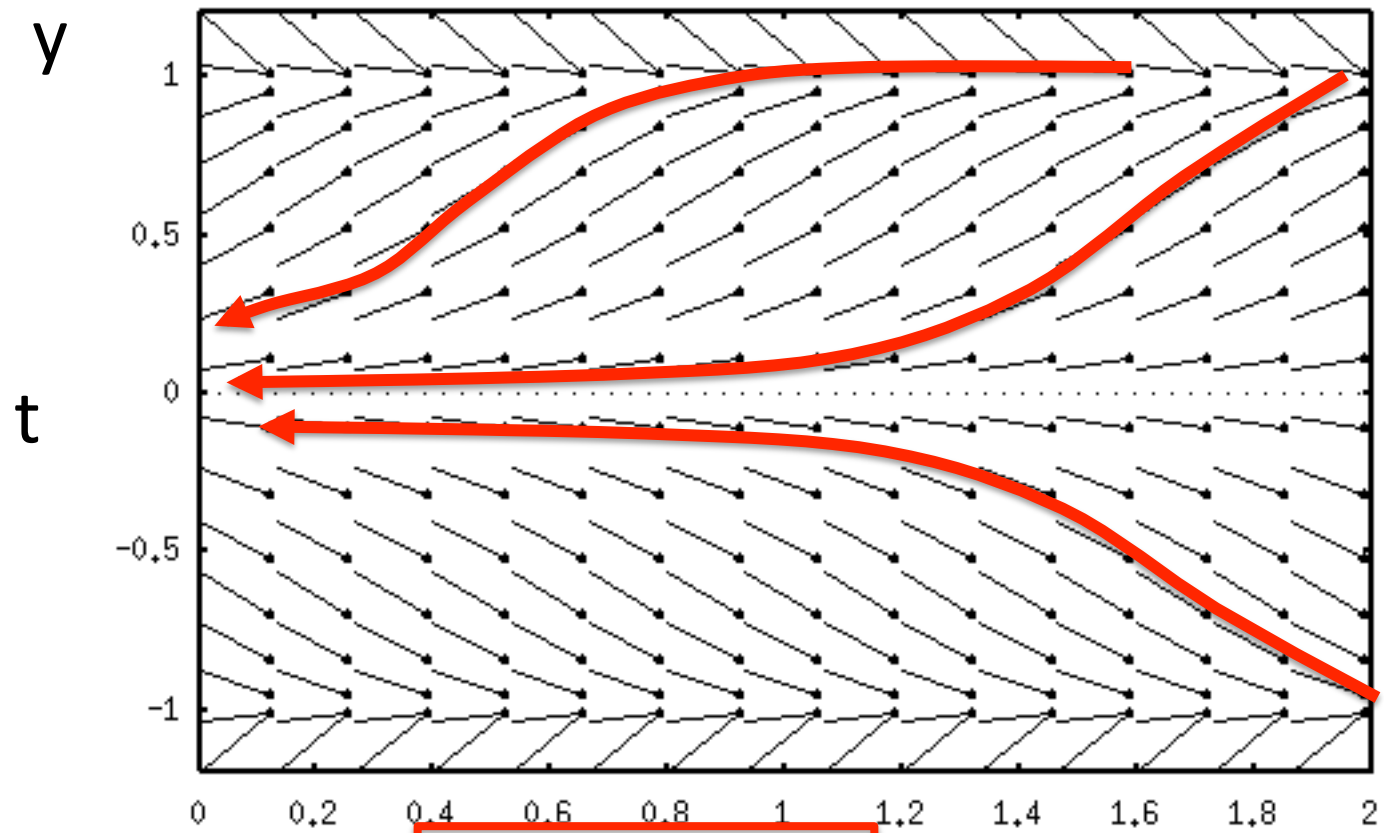

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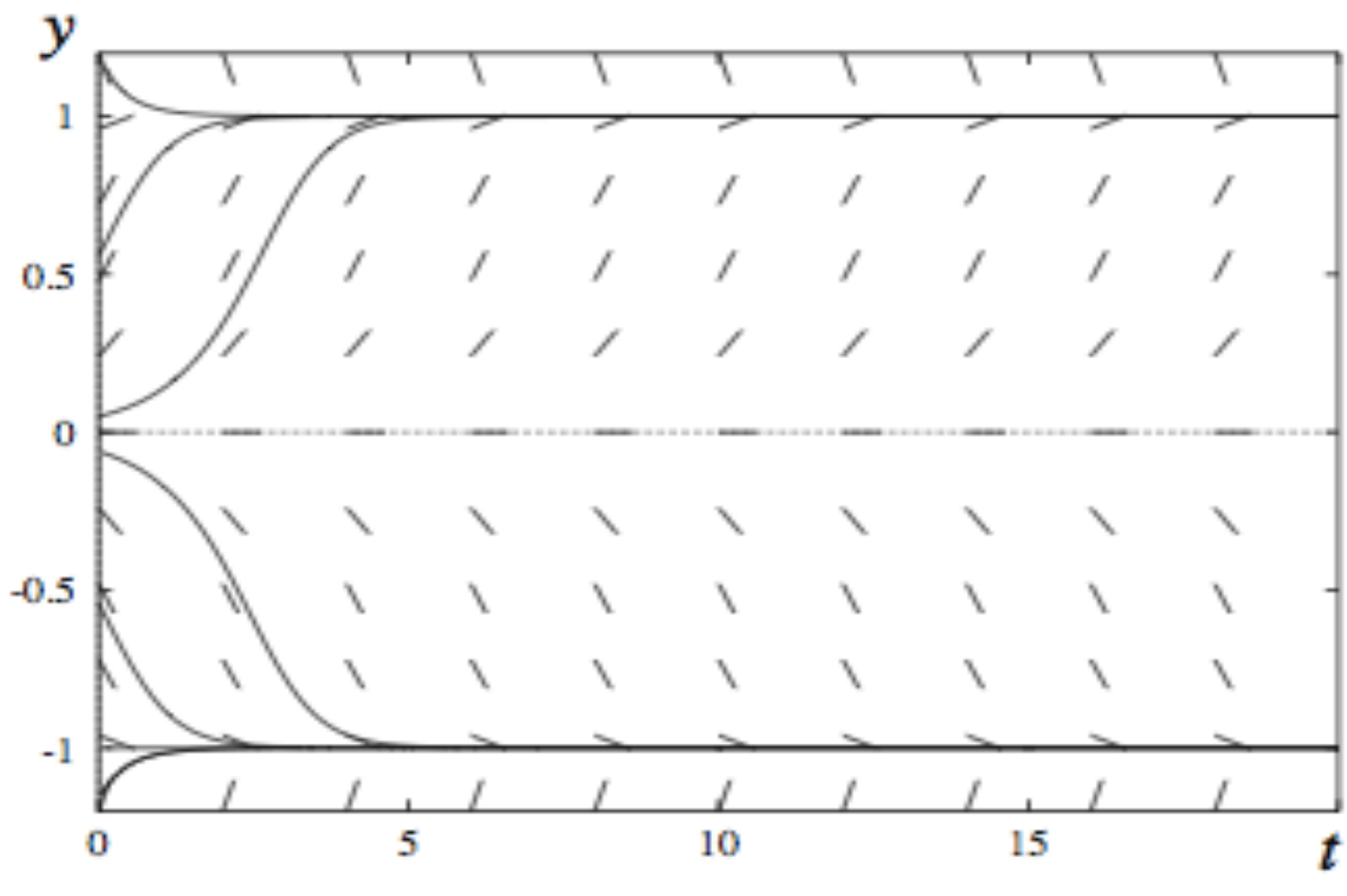
Solution look like this!

$$\frac{dy}{dt} = f(y) = y - y^3.$$



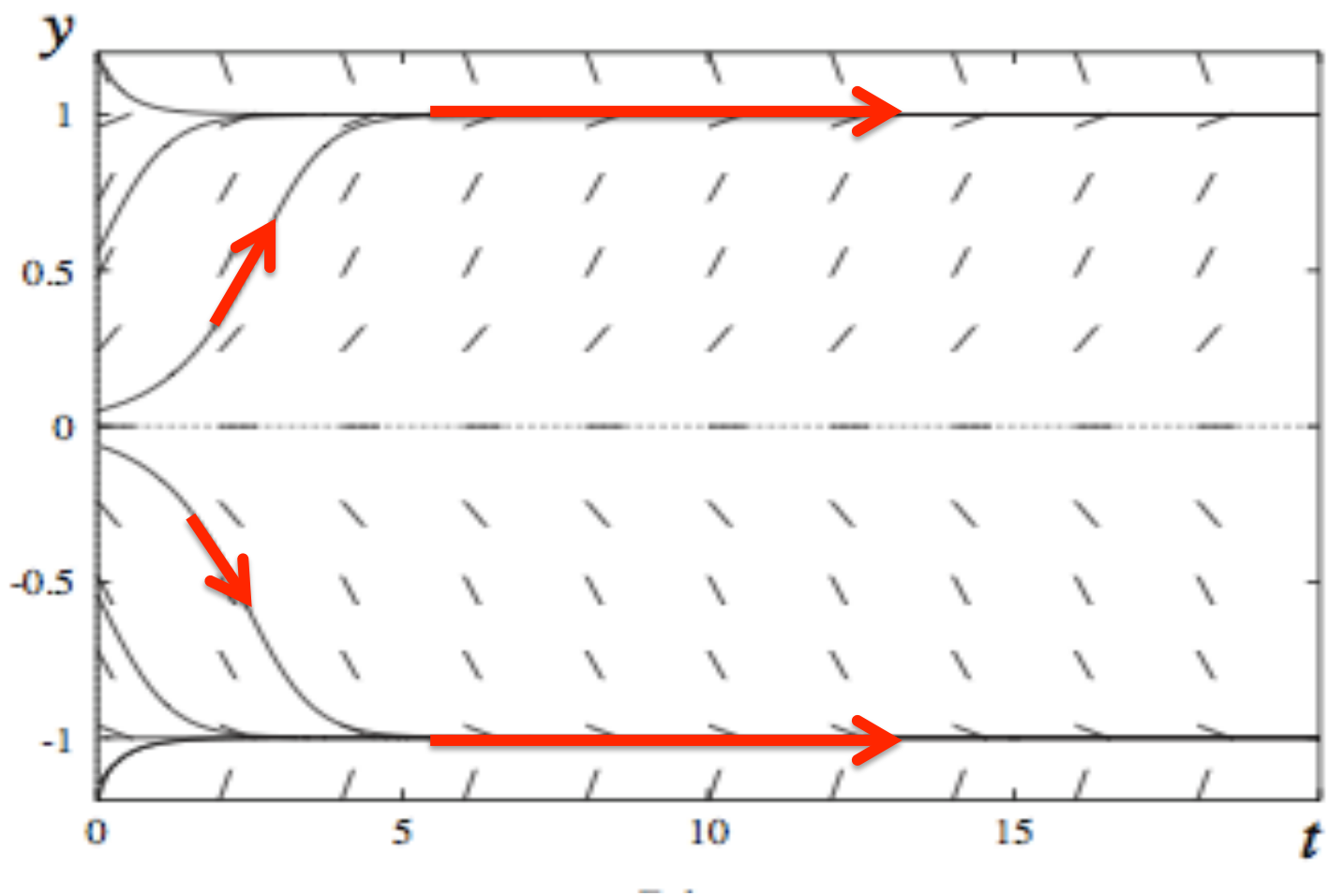
Rules for solution curves:

- Time is always flowing left to right

- Curves are parallel to the slopes at each point
- Curves do not cross
- Every initial condition corresponds to a unique curve.

Solution curves

$$\frac{dy}{dt} = f(y) = y - y^3.$$



$$\frac{dy}{dt} = f(y) = y - y^3.$$

Steady states

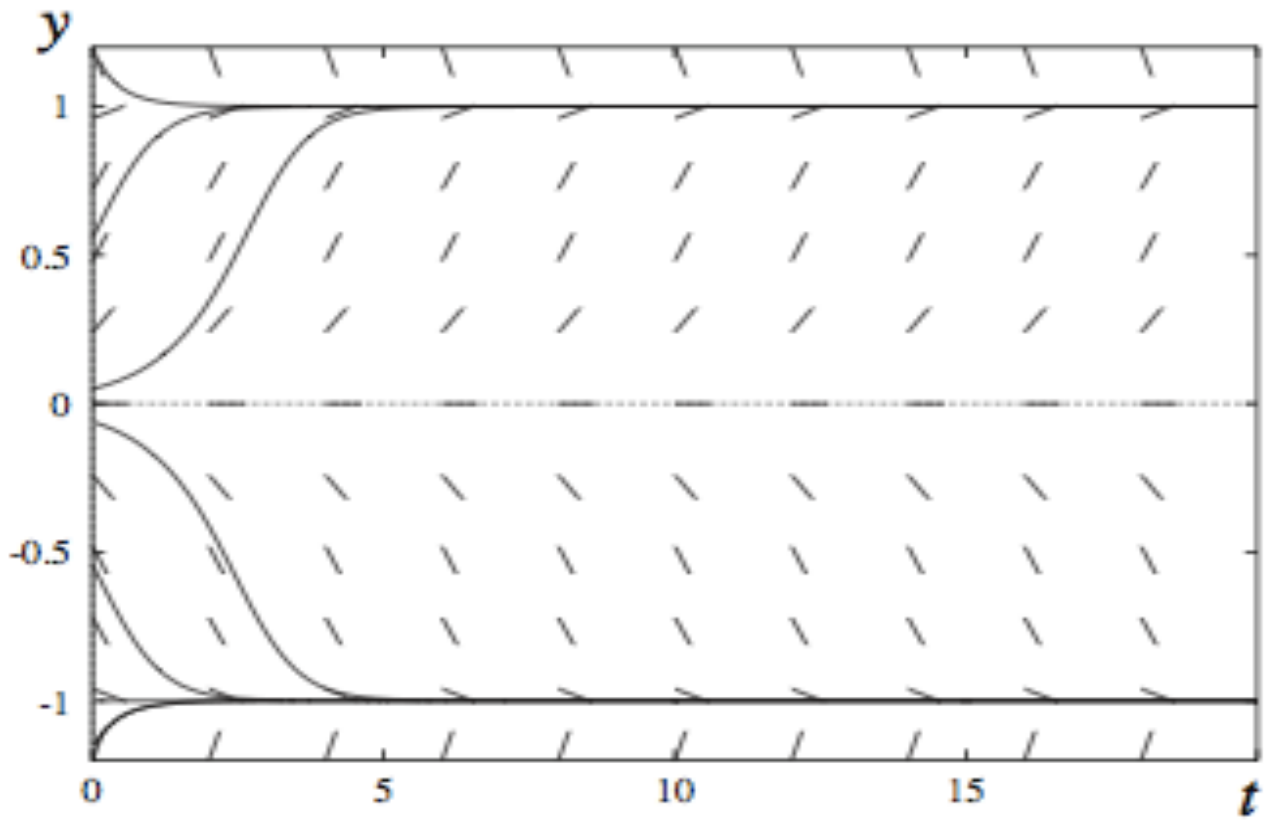
- A steady state is a state in which there is no change
- SS: $\frac{dy}{dt} = f(y) = 0$

- The steady states are y=-1, 0, 1

$$\frac{dy}{dt} = f(y) = y - y^3.$$

Stability

Definition 13.15 (Stability). *We say that a steady state is* **stable** *if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is* **unstable**, *if states that are initially very close to it eventually move away from that steady state.*



(6) The stable steady states are:

$$\frac{dy}{dt} = f(y) = y - y^3.$$

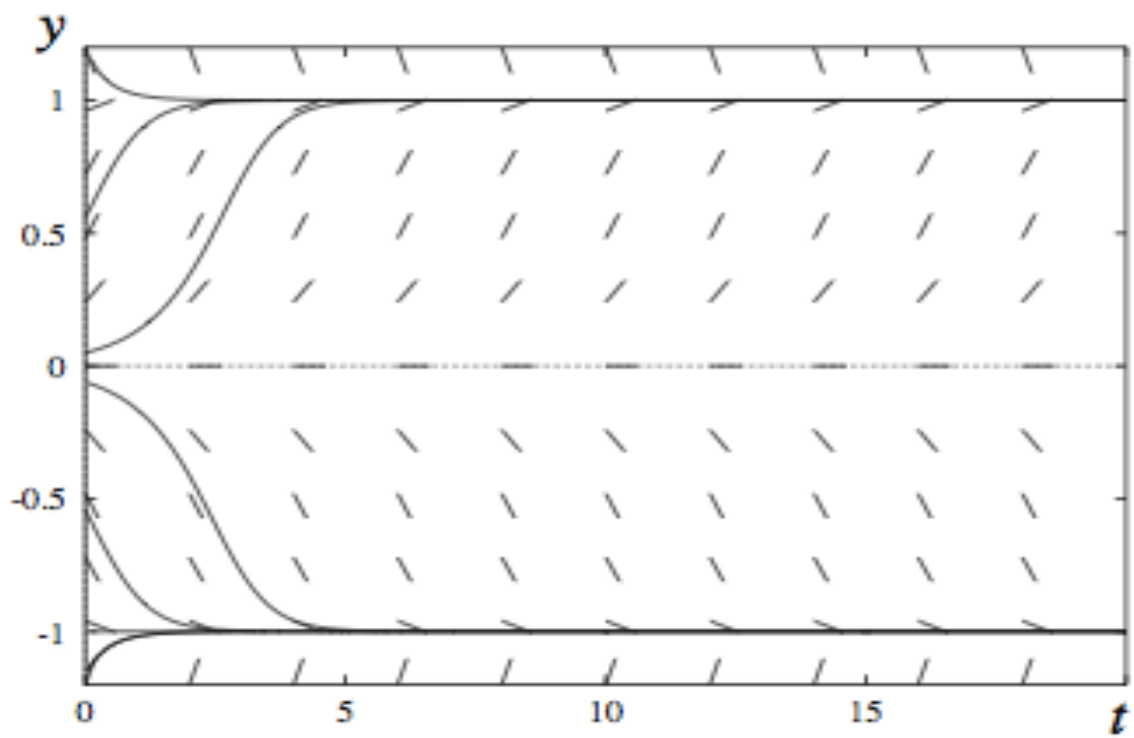
(A) y=0

(B) y=1

(C) y = -1, 1

(D) y=-1,0,1

(E) Not sure



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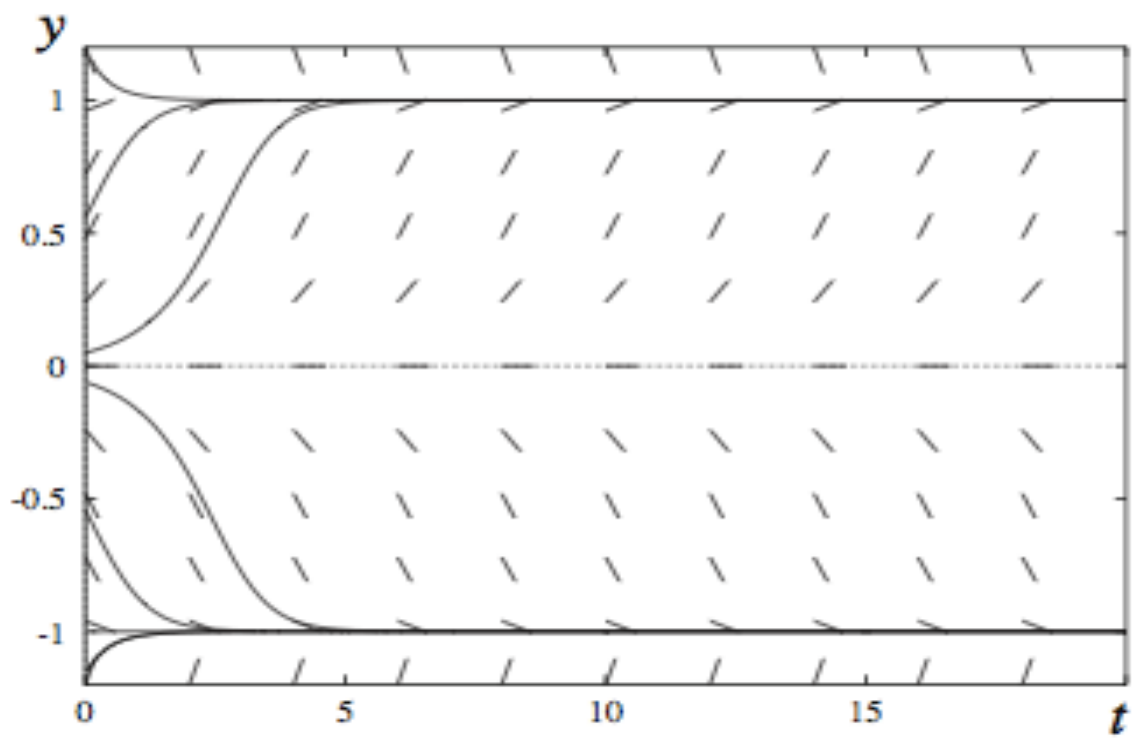

Stability

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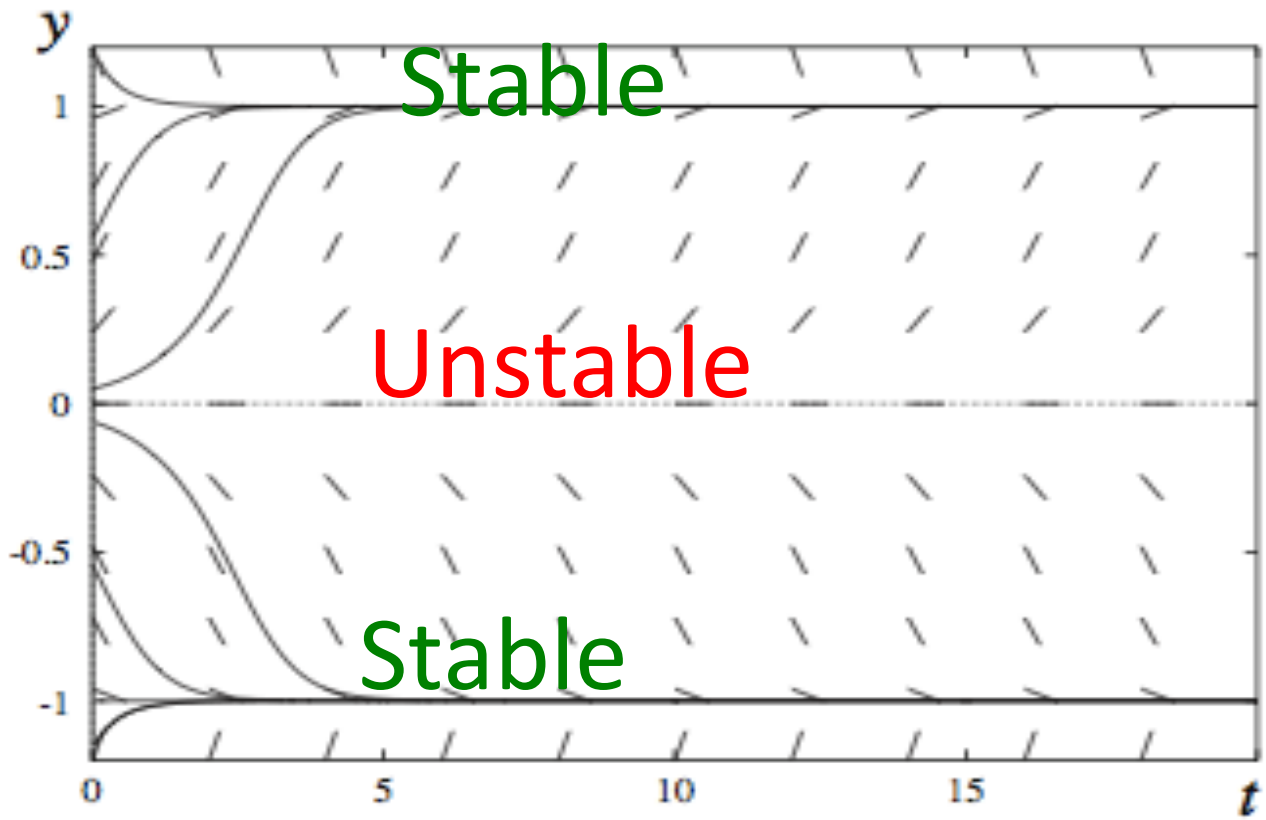


Problem-solving session

- See worksheet
- We worked out the full solution to: http://wiki.ubc.ca/Course:MATH102/Question_Challenge/2008_December_Q7

- We also worked on the cone in sphere problem (next slides)

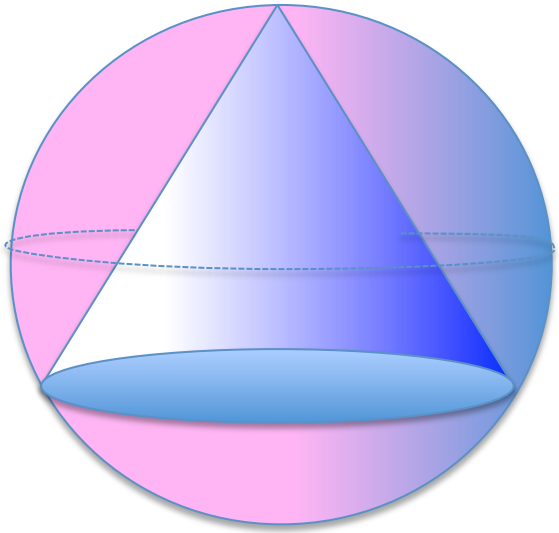
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius R.

The volume of a cone is

$V = (1/3)\ \pi\ r^2\ h$

where r is base radius and h is cone height

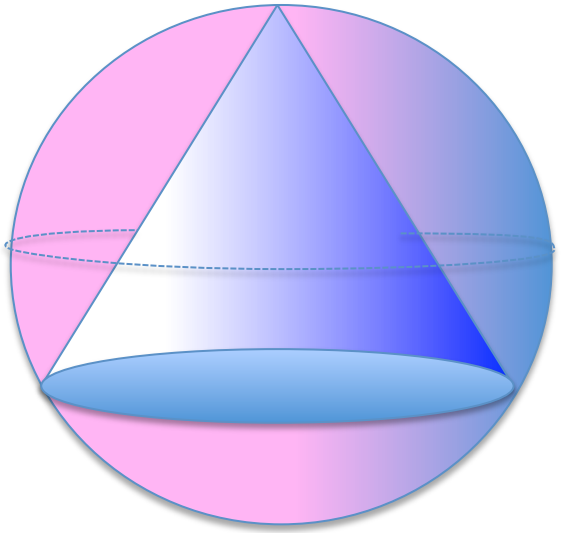


Cone inside a sphere

Goal: we want to maximize $V = (1/3)\,\pi\, r^2\, h$

But r and h have to be adjusted so that the cone fits inside the sphere.

This will impose a constraint.



What is that constraint?

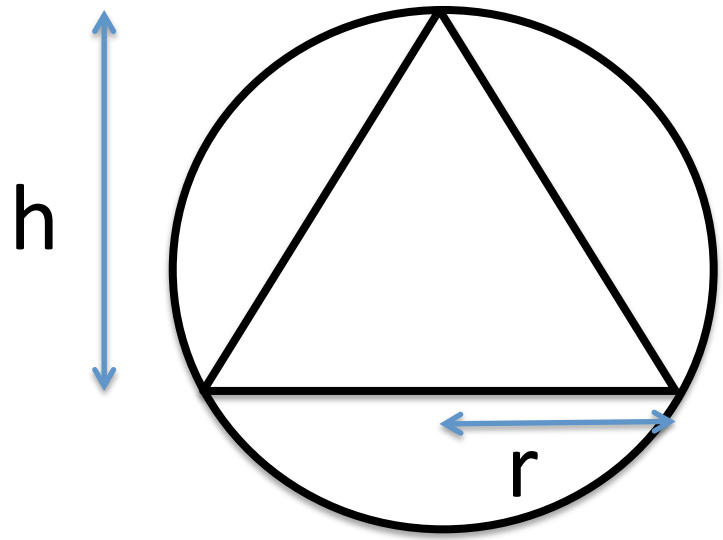
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius R.

Hint 1: slice the shapes

What do you see?

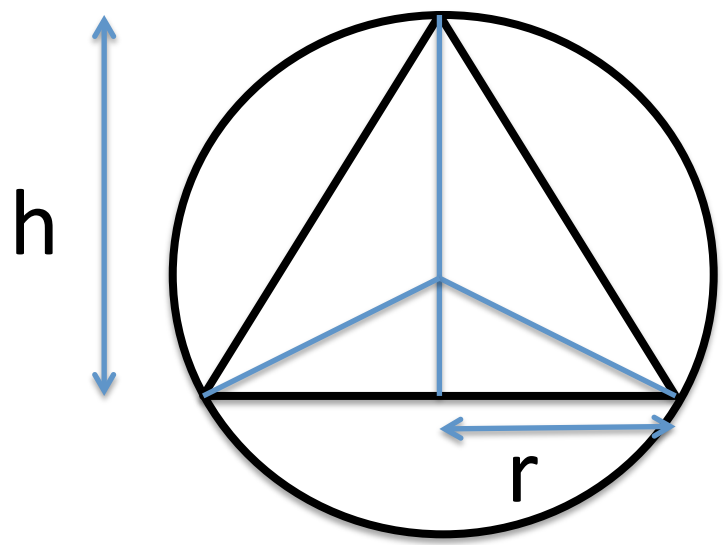
How is it related to R?



Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius R.

Hint 2: Put in some radii

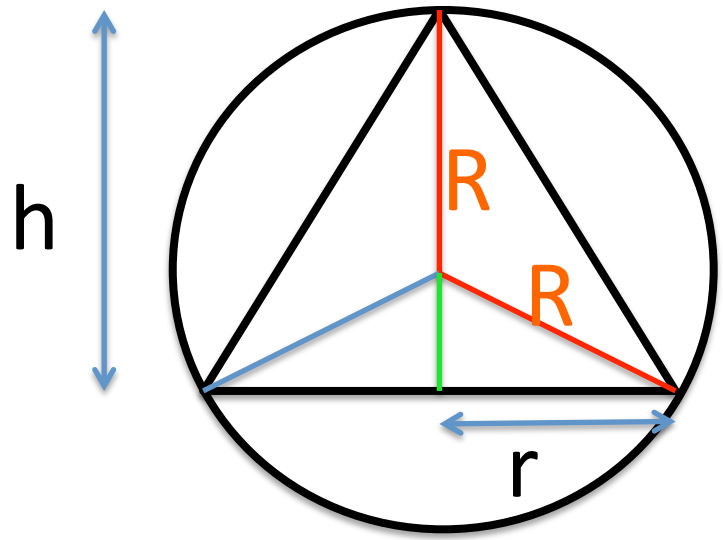


Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius R.

Hint 3: Add labels

What is this length?



Practice Exam Questions

From now on: see

http://wiki.ubc.ca/Course:MATH102/Question_Challenge