The geometry of change

Slope fields and other diagrams for qualitative solutions to differential equations
Help with Homework
Assignment 10:

For the solution that satisfies $y(0)=1$, sketch the solution curve and estimate $y(1) \approx$ and $y(-1) \approx$
Assignment10: Problem 8

For the solution that satisfies $y(0)=1$, sketch the solution curve and estimate $y(1) \approx$ and $y(-1) \approx$. 
The “math buddy” system

• Matchin up a few students who are comfortable with math with those who need some help.

• Let me know if you want to be a math buddy or you want to be matched up.

• Idea: someone to consult with quick questions when you are stuck.
More problems like on the exam

• Math 102 wiki
More problems like on the exam

• Our unique place to work and study online:
  • [http://wiki.ubc.ca/Course:MATH102/Question_Challenge](http://wiki.ubc.ca/Course:MATH102/Question_Challenge)

Questions in the Challenge Pool

• 2001 December Q03
• 2001 December Q5
• 2002 December Q6
• 2004 December Q6
• 2008 December Q2a
• 2008 December Q2b
The Math Exam Resource (MER)

- [http://wiki.ubc.ca/Science:Math_Exam_Resources/Courses/MATH102](http://wiki.ubc.ca/Science:Math_Exam_Resources/Courses/MATH102)

- Math 102 old exams: Hints and full solutions

Past Final Exams

- December 2014
- December 2013
- December 2012
- December 2011
Last time:

Differential equations and population growth
Unlimited population growth

So far we have one model for population growth. DE: \( \frac{dN}{dt} = rN \)

IC: \( N(0) = N_0 \)

Then \( N(t) = N_0 e^{rt} \)

Population grows EXPONENTIALLY

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Unlimited population growth

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Population grows EXPONENTIALLY

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Unlimited population growth

\[ \frac{dN}{dt} = rN \]

\[ N(0) = N_0 \rightarrow N(t) = N_0 e^{rt} \]

**Critique:**
1. Unrealistic growth
2. Resources limited
3. Not sustainable

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Checking

• Given a function, we can check if it satisfied a DE
• So we can check if it is a solution.
• Example: the DE \( \frac{dh}{dt} = -k\sqrt{h} \) represents the height of water draining out of a cylindrical container.
• The function \( h(t) = \left( \sqrt{h_0} - k\frac{t}{2} \right)^2 \) is a solution to that DE with \( h(0) = h_0 \).
• We can check that this is true
Finding

• But how would we find that function if we did not know it???

• Example: the DE \( \frac{dh}{dt} = -k\sqrt{h} \) represents the height of water draining out of a cylindrical container.

• The function \( h(t) = \left( \sqrt{h_0 - k\frac{t}{2}} \right)^2 \) is a solution to that DE with \( h(0) = h_0 \).

• We can check that this is true
The basic idea

• It is hard to find “formula” solutions to MOST differential equations.
• Sometimes we just want to know the overall (qualitative) behaviour..
• We can deduce some of this from properties of the differential equation itself? Without finding the full solution?
Example: Immigration in Europe

In Europe, birth rates are lower than mortality, but there is a constant rate of immigration, $I$

$$\frac{dN}{dt} = I - \mu N$$

where $\mu = m - r > 0$

$I > 0$

Constant rate of immigration

Constant rate of mortality

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Rate of change of population = rate immigration – rate mortality

\[
dN/dt = I - \mu N
\]

rate immigration

rate of mortality
Recall

\[
dN/dt = I - \mu N
\]

- This is a differential equation (DE). It keeps track of (instantaneous) changes.
- We are interested in finding a function of time \(N(t)\) that satisfies this equation and predicts the behaviour of the population over time.
- We call such a function a solution of the differential equation.
Special solutions

• In general, it is not easy to find the solution to a DE.

• But some solutions are “easy to find”

• For example: a solution in which there is NO CHANGE over time (where \( \frac{dN}{dt} = 0 \)).
Example: from last time

For what population level is there no change?

\[ \frac{dN}{dt} = I - \mu N \]

\[ \frac{dN}{dt} = 0 \]

No change
There is no change when immigration and mortality exactly balance

\[ 0 = \frac{dN}{dt} = I - \mu N \]

\[ I = \mu N \]

\[ N = \frac{I}{\mu} \]
There is no change when immigration and mortality exactly balance.

We say that

\[ N = \frac{1}{\mu} \]

is a steady state.
Steady state solution

- Population vs time:

\[ \frac{dN}{dt} = 1 - \mu N \]
What if immigration and mortality do not balance?

• How do we figure out what happens to the population over time?

• (intuitively: we expect N to either increase or decrease!)
Want to find a function $N(t)$

- That describes the behaviour of the population

\[
dN/dt = 1 - \mu N
\]
Want to find a function $N(t)$

- We might be able to find that function, but we can easily **SKETCH ITS GRAPH!!**

\[ \frac{dN}{dt} = 1 - \mu N \]
We will do so by thinking of slopes

Then DE describes slopes of $N$ vs $t$

$$\frac{dN}{dt} = I - \mu N$$

$N > \frac{I}{\mu}$  $\frac{dN}{dt} > 0$

$N = \frac{I}{\mu}$  $\frac{dN}{dt} = 0$

$N < \frac{I}{\mu}$  $\frac{dN}{dt} < 0$

This slide has been corrected from before
Put this on a graph $N$ vs $t$

- Indicate the slopes on a graph of $N$ vs $t$
If we start at the **steady state**, there is no change

- $N(0) = 1/\mu$
We could add more slopes

• This diagram is called a “slope field”
Slope field=tangent vectors

Solutions $N(t)$ has to keep these as tangents
Slope field = tangent vectors

Not allowed to “cross” any of those tangents

$N$

$I/\mu$

time, $t$
The logistic equation

A model for density-dependent growth
Population growth

So far we have one model for population growth.

Critique:
1. Unrealistic growth
2. Resources limited
3. Not sustainable
Revised “density dependent” growth

Hypotheses:
• Environment has limited resources to sustain population
• Growth cannot continue at constant per capita rate indefinitely
• Beyond some population size, there will be net mortality
Logistic growth

Similar idea: \[ \frac{dN}{dt} = gN \]

BUT now assume that growth rate \( g \) is not constant!

Assume that the per-capita growth rate decreases as population size increases.
“Density dependent” growth rate

• Assume $g(N)$ decreases linearly with $N$:
“Density dependent” per capita growth rate

- Assume $g(N)$ decreases linearly with $N$:

$$g_N = \text{"carrying Capacity"}$$
(4) The function $g(N)$ shown here is:

(A) $g(N) = rN - K$
(B) $g(N) = KN - r$
(C) $g(N) = (r-K)N$
(D) $g(N) = r(N-K)$
(E) $g(N) = r(K-N)/K$
(4) The function $g(N)$ shown here is:

(A) $g(N) = rN - K$

(B) $g(N) = KN - r$

(C) $g(N) = (r - K)N$

(D) $g(N) = r(N - K)$

(E) $g(N) = r(K - N)/K$
Solution:

\[ g = \frac{r}{\text{interc}} + \frac{1}{\text{slope}} \]

\[ = r \left( 1 - \frac{2r}{\kappa} \right) \]

\[ g(N) = r \left( \frac{\kappa - N}{\kappa} \right) \]
(5) So the new diff’l eqn we want is:

(A) \( \frac{dN}{dt} = rN(K-N)/K \)
(B) \( \frac{dN}{dt} = r(N-K)/K \)
(C) \( \frac{dN}{dt} = rN(N-K) \)
(D) \( \frac{dN}{dt} = rN \)
(E) Not sure
(5) So the new diff’l eqn we want is:

(A) \( \frac{dN}{dt} = \frac{rN(K-N)}{K} \)
(B) \( \frac{dN}{dt} = \frac{r(N-K)}{K} \)
(C) \( \frac{dN}{dt} = rN(N-K) \)
(D) \( \frac{dN}{dt} = rN \)
(E) Not sure
So the new diff’l eqn we want is:

(A) \(\frac{dN}{dt} = rN(K-N)/K\)

(B) \(\frac{dN}{dt} = r(K-N)/K\)

(C) \(\frac{dN}{dt} = rN(N-K)/K\)

(D) \(\frac{dN}{dt} = rN\)

(E) Not sure

\[
\frac{dN}{dt} = \left[ r \frac{(K-N)}{K} \right] N.
\]
### Population growth

<table>
<thead>
<tr>
<th>Unlimited growth</th>
<th>Density dependent growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dN}{dt} = rN )</td>
<td>( \frac{dN}{dt} = \left[ r \frac{(K - N)}{K} \right] N )</td>
</tr>
</tbody>
</table>

- **K** = carrying capacity
- **r** = intrinsic growth rate
Other ways of interpreting the logistic equation:

Expand:

\[
\frac{dN}{dt} = rN - \frac{r}{K}N^2
\]

Simple birth term (just like before)

Mortality when overcrowded
Create a table and draw slope field for logistic equation

\[ \frac{dN}{dt} = \left[ r \left( \frac{K - N}{K} \right) \right] N. \]

Fill in this table:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{dN}{dt} )</th>
<th>behaviour</th>
<th>tangent line</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>---</td>
<td>→</td>
</tr>
<tr>
<td>0&lt;N&lt;K</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N=K</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N&gt;K</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Solution:

\[ \frac{dN}{dt} = \left[ r \frac{(K - N)}{K} \right] N. \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{dN}{dt} )</th>
<th>beh.</th>
<th>targ.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \rightarrow )</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>( 0 &lt; N &lt; K )</td>
<td>+</td>
<td>incr.</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>( K )</td>
<td>0</td>
<td>( \rightarrow )</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>( N &gt; K )</td>
<td>-ve</td>
<td>decay</td>
<td>( \rightarrow )</td>
</tr>
</tbody>
</table>
(6) Slope field (logistic)

\[
\frac{dN}{dt} = \left[ \frac{r(K - N)}{K} \right] N.
\]

Which plot?

- **A**
- **B**
- **C**
- **D**
(6) Slope field (logistic) 

Which plot?

\[
\frac{dN}{dt} = \left[ \frac{r(K-N)}{K} \right] N.
\]
Solution curves

Use your slope field to draw a few solution curves \( N(t) \) to the logistic equation.

Then interpret the behaviour in plain English.

Explain what happens to the population after a long time.
Slope field and some solutions

Challenge: show that only solution curves that start with $0<N<K/2$ have an inflection point.
Experiments

G.F. Gause November, 1934
“The struggle for existence”
http://www.ggause.com/Contgau.htm

Fig. 4. The growth of population of *Paramecium caudatum*
Example:

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

• Use the same kind of geometric (qualitative) approach.
\[
\frac{dy}{dt} = f(y) = y - y^3.
\]

<table>
<thead>
<tr>
<th>y</th>
<th>dy/dt</th>
<th>behaviour</th>
<th>tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tabulate values

\[
\frac{dy}{dt} = f(y) = y - y^3.
\]

<table>
<thead>
<tr>
<th>$y$</th>
<th>sign of $f(y) = y - y^3$</th>
<th>behaviour of $y$</th>
<th>direction of arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y &lt; -1$</td>
<td>+ve</td>
<td>increasing</td>
<td>↑</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>no change in $y$</td>
<td>→</td>
</tr>
<tr>
<td>-0.5</td>
<td>-ve</td>
<td>decreasing</td>
<td>↓</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>no change in $y$</td>
<td>→</td>
</tr>
<tr>
<td>0.5</td>
<td>+ve</td>
<td>increasing</td>
<td>↑</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>no change in $y$</td>
<td>→</td>
</tr>
<tr>
<td>$y &gt; 1$</td>
<td>-ve</td>
<td>decreasing</td>
<td>↓</td>
</tr>
</tbody>
</table>

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How many steady states?

• (A) None
• (B) One
• (C) Two
• (D) three
• (E) Not sure
How many steady states?

- (A) None
- (B) One
- (C) Two
- (D) three
- (E) Not sure

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

\[ 0 = y - y^3 \]
\[ 0 = y (1 - y^2) \]
So \( y = 0, 1, -1 \)
(2) The slope field should look like:

\[ \frac{dy}{dt} = f(y) = y - y^3. \]
(2) The slope field should look like:

\[
\frac{dy}{dt} = f(y) = y - y^3.
\]
How do we know?

For small $y$, the DE is roughly $\frac{dy}{dt} = y$ so $y$ increasing
(3) Solutions look like this?

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

(A) TRUE  (B) FALSE  (C) Not sure
(3) Solutions look like this?

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

(A) TRUE  (B) FALSE  (C) Not sure
(4) Solutions look like this?

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

(A) TRUE  (B) FALSE  (C) Not sure
(4) Solutions look like this?

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

- (A) TRUE
- (B) FALSE
- (C) Not sure
Solution look like this!

\[
\frac{dy}{dt} = f(y) = y - y^3.
\]
Rules for solution curves:

- Time is always flowing left to right
- Curves are parallel to the slopes at each point
- Curves do not cross
- Every initial condition corresponds to a unique curve.
Solution curves

\[ \frac{dy}{dt} = f(y) = y - y^3. \]
Steady states

- A steady state is a state in which there is no change
- SS: \[ \frac{dy}{dt} = f(y) = y - y^3. \]

- The steady states are \( y = -1, 0, 1 \)
Stability

\[ \frac{dy}{dt} = f(y) = y - y^3. \]

**Definition 13.15 (Stability).** We say that a steady state is **stable** if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is **unstable**, if states that are initially very close to it eventually move away from that steady state.
(6) The stable steady states are:

(A) $y = 0$
(B) $y = 1$
(C) $y = -1, 1$
(D) $y = -1, 0, 1$
(E) Not sure
(6) The stable steady states are:

(A) $y = 0$
(B) $y = 1$
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Definition 13.15 (Stability). We say that a steady state is **stable** if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is **unstable**, if states that are initially very close to it eventually move away from that steady state.
Problem-solving session

• See worksheet

• We worked out the full solution to:
  http://wiki.ubc.ca/Course:MATH102/Question_Challenge/2008_December_Q7

• We also worked on the cone in sphere problem (next slides)
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius $R$.

The volume of a cone is

$$V = \left( \frac{1}{3} \right) \pi r^2 h$$

where $r$ is base radius and $h$ is cone height.
Cone inside a sphere

Goal: we want to maximize \( V = \frac{1}{3} \pi r^2 h \)

But \( r \) and \( h \) have to be adjusted so that the cone fits inside the sphere.

This will impose a constraint.

What is that constraint?
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius $R$.

Hint 1: slice the shapes

What do you see?

How is it related to $R$?
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius $R$.

Hint 2: Put in some radii
Cone inside a sphere

Find the dimensions of the cone with largest volume that fits inside a sphere of radius $R$.

Hint 3: Add labels

What is this length?
Practice Exam Questions

From now on: see

http://wiki.ubc.ca/Course:MATH102/Question_Challenge