AN INTERESTING 2-PERIODIC ALTERNATING KNOT

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ABSTRACT. In this note we give an example of an alternating 2-periodic knot with signature -4, but which has a quotient knot with signature 0. This is a counterexample to a conjectural filtered rank inequality for the knot Floer homology of periodic knots.

In [Boy18], there is the following conjecture.

Conjecture 1. *[Boy18, Conjecture 1] Let $\widetilde{K}$ be a 2-periodic knot in S^3 with quotient knot K and axis A, and let λ be $lk(K, A)$. Then*

$$\sum_{i \geq q} \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, \widetilde{a}) \oplus \widehat{HFK}_i(\widetilde{K}, \widetilde{a}+1)\right) \geq \sum_{2i \geq q+1} \operatorname{rank} \widehat{HFK}_i(K, a),$$

where $\widetilde{a} = 2a + \dfrac{\lambda - 1}{2}$.

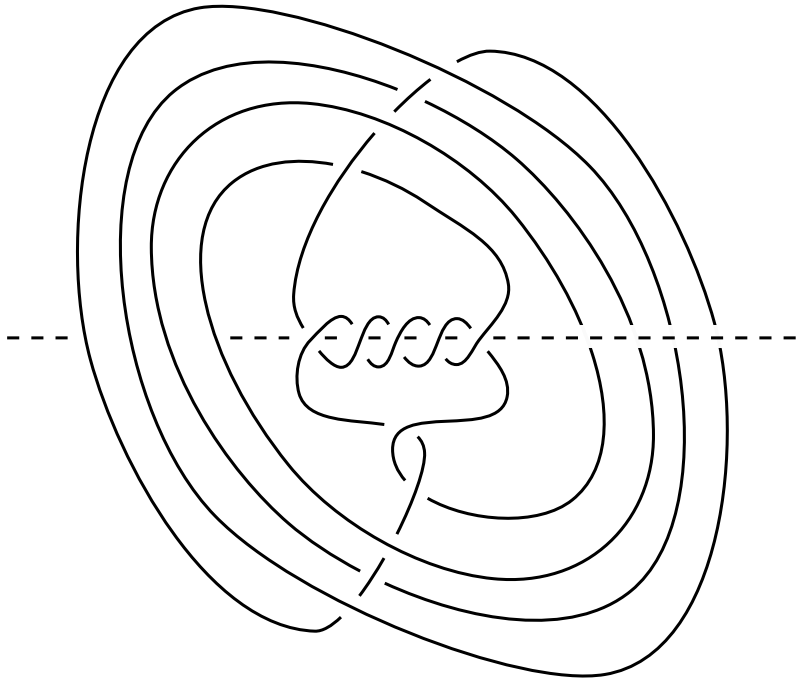

FIGURE 1. The 2-bridge knot $K_{127/217}$ is alternating and 2-periodic. The 2-period can be seen diagrammatically by performing a flype on the center 5 crossings, then rotating by π within the plane of the diagram. The dotted line is the axis of symmetry, which passes through the center 5 crossings.

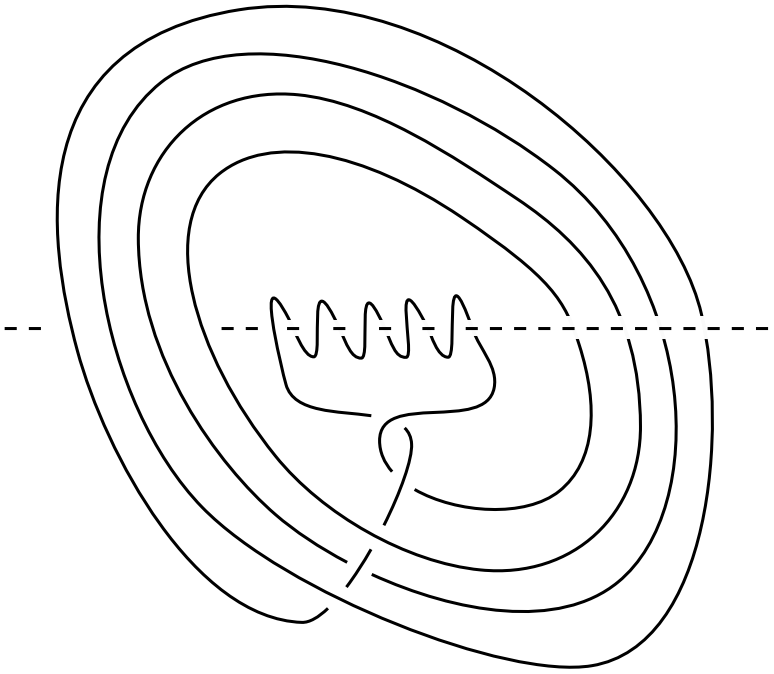

FIGURE 2. The quotient of the knot in Figure 1 is the unknot.

This conjecture implies the following conjecture for alternating knots; see [Boy18, Theorem 6] and [Boy21, Theorem 1.1].

Conjecture 2. *[Boy18, Theorem 6] Let $\widetilde{K}$ be a 2-periodic alternating knot in S^3 with quotient K and having linking number λ with the axis. Then*

$$|2\sigma(K) - \sigma(\widetilde{K})| \leq \lambda + 1.$$

However, consider the 2-periodic knot $\widetilde{K}$ shown in Figure 1, which has the quotient knot K shown in Figure 2. One readily computes that $\sigma(\widetilde{K}) = -4$, $\lambda = 1$, and $\sigma(K) = 0$. However, $|0 + 4| > 2$ so that Conjecture 2, and hence Conjecture 1, is false.

This example relies on a 2-periodic alternating knot which has no 2-periodic alternating diagram. In other words if we force the symmetry of $\widetilde{K}$ to be rotation around an axis perpendicular to the plane of the diagram, then the diagram will no longer be alternating. However, every p-periodic alternating knot with $p > 2$ has a p-periodic alternating diagram (see [CQH21] and [Boy19]) so that our counterexample does not generalize to rule out a version of these conjectures for $p > 2$.

REFERENCES

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