Exponential growth and decay

The differential equation

$$\frac{dy}{dt} = ky$$

with initial condition $y(0) = y_0$ has the solution

$$y(t) = y_0 e^{kt}.$$

When $k > 0$, then we say that y grows exponentially, and when $k < 0$, then we say y decays exponentially.

Here are some examples of exponential growth:

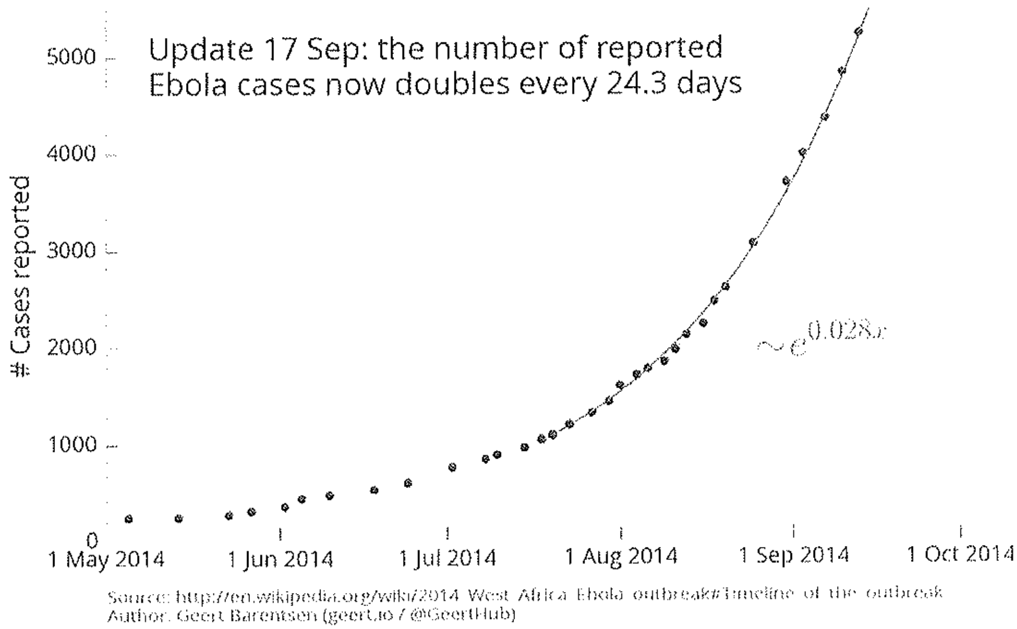
- Population growth with unlimited resources (for example, bacteria).
- Spread of a virus.
- Bank account with continuously compounded interest.

Examples of exponential decay:

- Amount of radioactive material.
- Temperature of a cooling object.
- Clearing of a drug from an organism.

The **half-life** of a radioactive isotope is the length of time after which only half of the original amount is left. Half-life can be determined from the constant k. It does not depend on the original amount y_0.

PROBLEM 1. **Ebola epidemic.** A UK researcher Geert Barentsen has taken the number of reported Ebola cases worldwide and fitted an exponential curve to the data. This suggests that the number of infected people grows exponentially with $k = 0.028$



1. Verify the claim that with $k = 0.028$, the number of infected people doubles every 24.3 days.
2. Assuming that $y(0) = 1000$ in July 15, and the number of infected people grows exponentially at the same rate, when will all 7 billion people on Earth be infected?

1. Find x, such that

$$y_0 \cdot e^{0.028x} = 2y_0 \implies 0.028x = \ln 2, \quad x = \frac{\ln 2}{0.028} \approx 24$$

2. Find x, such that

$$y_0 \cdot e^{0.028x} = 7,000,000,000 \qquad y_0 = 1000$$

$$0.028x = \ln 7,000,000$$

$$x = \frac{\ln 7,000,000}{0.028}$$

PROBLEM 2. **Radiocarbon dating.** Every living organism on Earth contains the same ratio of the radioactive carbon isotope carbon-14 to the more common carbon-12 isotope. (This is because the Earth's atmosphere naturally produces carbon-14 and keeps it at a constant level.) When an organism dies, the radioactive carbon-14 starts decaying exponentially, with half-life approximately 5700 years. Measuring the remaining amount of carbon-14 in bones, coal, paint on cave walls, etc, we can determine how old these are.

1. A painting contains 95% of the original amount of carbon-14. Determine the age of the paining.
2. Radiocarbon dating with carbon-14 does not work well if the sample is older than about 50,000 years because there is very little carbon-14 left. How much carbon-14 is left in a 50,000 year old mammoth bone? Give your answer as a percentage of the original amount.

First use $\frac{1}{2}$-life to find k:

$$e^{k \cdot 5700} = \frac{1}{2} \implies k = \frac{-\ln 2}{5700}.$$

1. Find $\cancel{t}$, t, such that

$$e^{kt} = .95 \implies t = \frac{\ln .95}{k} = \frac{\ln .95}{\ln \frac{1}{2}} \cdot 5700$$

2. Find p, such that

$$e^{k \cdot 50{,}000} = p$$

PROBLEM 3. **Newton's law of cooling.** Let $T(t)$ be the temperature of an object. Then T approaches the surrounding temperature T_s at an exponential rate. In other words, the difference $T(t) - T_s$ decays exponentially:

$$T(t) - T_s = (T_0 - T_s)e^{-kt}.$$

1. A bowl of soup cools from 90° to 70° in 10 minutes in the room temperature 20°. How long does it take for the soup to cool from 70° to 50°?

2. A turkey is put in the oven. It heats from 20° to 100° in 10 minutes and then from 100° to 120° in another 10 minutes. What is the temperature of the oven? (Hint: write down the two equations with unknowns T_s and k. Solve for e^{-10k} in both equations and set them equal. Then solve for T_s.)

1. Find k from

$$70 - 20 = (90-20)\,e^{-k\cdot 10} \Rightarrow k = -\frac{1}{10}\ln\frac{50}{70}$$

Now find t, such that

$$50 - 20 = (70-20)\,e^{-k\cdot t} \Rightarrow t = -\frac{1}{k}\ln\frac{30}{50}$$

2. Unknowns: k and T_s,

$$\begin{cases} 100 - T_s = (20 - T_s)\,e^{-k\cdot 10} \\ 120 - T_s = (100 - T_s)\,e^{-k\cdot 10} \end{cases}$$

Solve for $e^{-k\cdot 10}$, set equal:

$$\frac{100 - T_s}{20 - T_s} = \frac{120 - T_s}{100 - T_s}$$

$$(100 - T_s)^2 = (20 - T_s)(120 - T_s)$$

$$100^2 - 200T_s + T_s^2 = 20\cdot 120 - 140\,T_s + T_s^2$$

$$10{,}000 - 2{,}400 = 60\,T_s$$

$$T_s = \frac{7600}{60} = \frac{760}{6} = 127$$

PROBLEM 4. **Population growth with harvesting.** Let $P(t)$ be the population of a fishery (in thousands of fish). Assume that $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = 0.1P - t^2.$$

The first term on the right tells us that the population grows exponentially. The second term describes harvesting: in year t we catch t^2 thousand fish.

1. Find a solution to the differential equation of the form $P(t) = at^2 + bt + c$. (Hint: substitute this polynomial $P(t)$ into the differential equation above and solve for a, b, c. You should get an equation polynomial=polynomial; use the fact that two polynomials are equal if and only if their coefficients are equal.)

2. Show that it $P(t)$ is one solution to the differential equation, then $P(t) + De^{0.1t}$ is also a solution for any constant D. (Substitute $P(t) + De^{0.1t}$ into the differential equation and check that equality holds.)

3. For an initial condition $P(0) = P_0$, find the solution $P(t)$. Determine which initial populations P_0 grow to infinity and which initial populations eventually die out. (Think of P_0 as a given number and find D, such that $P(t) + De^{0.1t}|_{t=0} = P_0$.)

1. Substitute at^2+bt+c into equation:

$$(at^2+bt+c)' = 0.1(at^2+bt+c) - t^2$$

$$2at + b = (0.1a - 1)t^2 + 0.1b + 0.1c$$

Equate coefficients:

$$\begin{cases} 0 = 0.1a - 1 \\ 2a = 0.1b \\ b = 0.1c \end{cases} \qquad \begin{aligned} a &= 10 \\ b &= 200 \\ c &= 2000 \end{aligned} \qquad \Rightarrow \quad 10t^2 + 200t + 2000$$

2. Substitute $P(t) + De^{0.1t}$ into equation:

$$(P(t) + De^{0.1t})' = 0.1(P(t) + De^{0.1t}) - t^2$$ check this ✓

3. $$P(t) = 10t^2 + 200t + 2000 + De^{0.1t}$$

$$P(0) = 2000 + D = P_0 \qquad \Rightarrow \qquad D = P_0 - 2000.$$

$$P(t) \xrightarrow[t\to\infty]{} \begin{cases} \infty & \text{if } D \geq 0 \iff P_0 \geq 2000 \\ -\infty & \text{if } D < 0 \iff P_0 < 2000. \end{cases}$$