SCIENCE ONE – MATHEMATICS
MIDTERM PRACTICE EXAM
OCTOBER, 2014

Name:
Student ID:

Exam rules:

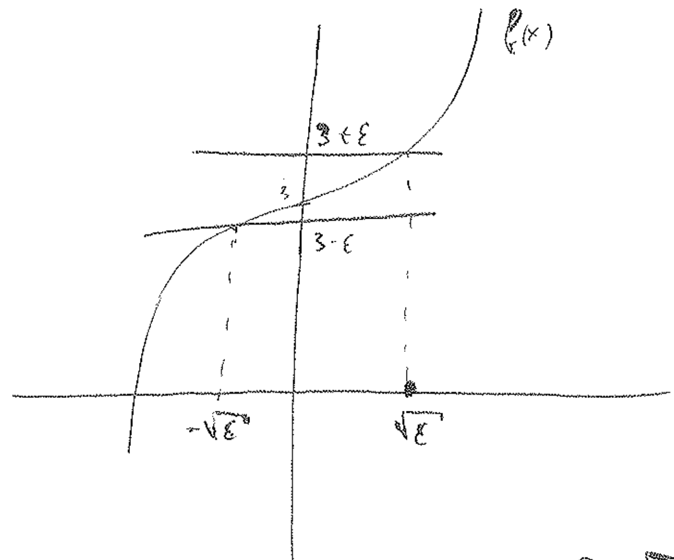
- No calculators, other electronic devices, open books or notes are allowed.
- There are 7 problems in this exam. Use opposite empty pages if needed.

1		10
2		10
3		6
4		10
5		10
6		10
7		10
Total		66

PROBLEM 1. [10 pts.] Use the $\varepsilon - \delta$ definition of limits to prove that

$$f(x) = x \cdot |x| + 3$$

is continuous at $x = 0$.



From the picture, we can determine that $\delta = \sqrt{\varepsilon}$

We need to prove:

$$\lim_{x \to 0} f(x) = f(0) = 3.$$

Let $\varepsilon > 0$ be arbitrary. Take $\delta = \sqrt{\varepsilon}$.

If $0 < |x - 0| < \delta$, that means $0 < |x| < \delta$, then

$$|f(x) - 3| = |x \cdot |x|| = |x| \cdot |x| < \delta \cdot \delta = \varepsilon.$$

PROBLEM 2. [10 pts.]Let

$$f(x) = \frac{x^2+1}{x-1}.$$

Find the derivative of $f(x)$ at an arbitrary point $x = a$ using the definition of the derivative as a limit. No marks will be given for the use of differentiation rules.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \left[\frac{(a+h)^2+1}{a+h-1} - \frac{a^2+1}{a-1} \right] \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{[(a+h)^2+1](a-1) - (a+h-1)(a^2+1)}{(a+h-1)(a-1)h}$$

$$= \lim_{h \to 0} \frac{a^3+2a^2h+ah^2+a-a^2-2ah-h^2-1-a^3-a^2h+a^2-a-h+1}{(a+h-1)(a-1)h}$$

$$= \lim_{h \to 0} \frac{a^2h+ah^2-2ah-h^2-h}{(a+h-1)(a-1)h}$$

$$= \lim_{h \to 0} \frac{a^2+ah-2a-h-1}{(a+h-1)(a-1)} = \frac{a^2-2a-1}{(a-1)(a-1)}.$$

PROBLEM 3. [6 pts.] Prove that the equation $2^x + 3^x = 4^x$ has a solution $x > 0$. (You may assume that for any $a > 0$, the function $f(x) = a^x$ is continuous.)

Let $f(x) = 2^x + 3^x - 4^x$.

Then

$$f(1) = 2 + 3 - 4 = 1$$

$$f(2) = 4 + 9 - 16 = -3$$

IVT $\Rightarrow$ $f(x) = 0$ for some x in $(1, 2)$.

PROBLEM 4. Use limit rules to find the following limits. If the limit does not exist, explain why. No $\varepsilon - \delta$ proofs are required.

1.[5 pts.]

$$\lim_{x\to\infty}(x - \sqrt{x^2+4x}).$$

$$\left(x - \sqrt{x^2+4x}\right)\cdot\frac{\left(x+\sqrt{x^2+4x}\right)}{x+\sqrt{x^2+4x}} =$$

$$= \frac{\cancel{x^2} - \cancel{x^2} - 4x}{x+\sqrt{x^2+4x}} = \frac{-4x}{x+\sqrt{x^2+4x}}\begin{matrix}\cdot\frac{1}{x}\\ \cdot\frac{1}{x}\end{matrix}$$

$$= \frac{-4}{1+\sqrt{1+\frac{4}{x}}} \xrightarrow[x\to\infty]{} \frac{-4}{1+1} = -2$$

2.[5 pts.]

$$\lim_{x\to 0}\frac{\tan x^2 + x\cos 3x}{\sin 5x\cos x}.$$

$$\frac{\tan x^2 + x\cos 3x}{\sin 5x\cos x}\begin{matrix}\cdot\frac{1}{\cancel{x}}\\ \cdot\frac{1}{x}\end{matrix} = \frac{\underbrace{\frac{\tan x^2}{x^2}}_{\xrightarrow{x\to 0}1}\cdot \underbrace{\cancel{x}}_{\to 0} + \underbrace{\cos(3x)}_{\to 1}}{\underbrace{\frac{\sin 5x}{5x}}_{\xrightarrow{x\to 0}1}\cdot 5\cdot \underbrace{\cos(x)}_{\to 1}}$$

$$\xrightarrow[x\to 0]{} \frac{0+1}{5} = \frac{1}{5}$$

PROBLEM 5. Use derivation rules.

1.[5 pts.] Find all points (x, y), such that the curve defined by $y = x^{\sqrt{x}}$ has a horizontal tangent line at (x, y). (Note that $y = x^{\sqrt{x}}$ is defined for $x > 0$ only, hence also $y > 0$.)

$$\ln y = \sqrt{x} \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx} \sqrt{x} \ln x$$

$$\frac{1}{y} \cdot y' = \frac{1}{2\sqrt{x}} \cdot \ln x + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right)$$

$$y' = y \cdot \frac{1}{\sqrt{x}} \cdot \left(\frac{1}{2}\ln x + 1\right)$$

Because $y > 0$, $x > 0$, $y' = 0$ if and only if

$$\frac{1}{2}\ln x + 1 = 0$$

$$\ln x = -2$$

$$x = e^{-2}, \quad y = \left(e^{-2}\right)^{e^{-1}} = e^{-\frac{2}{e}}$$

2.[5 pts.] The line $y = 2x - 3$ is tangent to the graph of $f(x) = x^2 + Bx + 1$ at some point P. Find all possible values for the constant B.

Let the point P be (x, y). Then

1. P lies on both curves:

$$2x - 3 = x^2 + Bx + 1$$

2. Slopes at P are equal

$$2 = 2x + B$$

Solve for B from

$$\begin{cases} 2x - 3 = x^2 + Bx + 1 \\ 2 = 2x + B \end{cases}$$

(2) $\Rightarrow$ $x = 1 - \frac{B}{2}$

(1) $2 - B - 3 = \left(1 - \frac{B}{2}\right)^2 + B\left(1 - \frac{B}{2}\right) + 1$

$$\frac{B^2}{4} - B - 3 = 0$$

$$B^2 - 4B - 12 = 0$$

$$B = 2 \pm \sqrt{4 + 12} = 2 \pm 4$$

$B = -2$ or $B = 6$

PROBLEM 6. [10 pts.] Find all values a, b, c, d, such that the function

$$f(x) = \begin{cases} 2x + a & \text{if } x < 0 \\ x^2 + bx + c & \text{if } 0 \leq x \leq 1 \\ dx - 2 & \text{if } x > 1 \end{cases}$$

is continuous and differentiable everywhere.

values are equal at 0, 1:

$$\begin{cases} a = c \\ 1 + b + c = d - 2 \end{cases}$$

slopes are equal at 0, 1:

$$\begin{cases} 2 = b \\ 2 + b = d \end{cases}$$

Solve:

$b = 2, \quad d = 4, \quad c = -1, \quad a = -1$

PROBLEM 7. [10pts.] A population is modeled by the differential equation

$$\frac{dP}{dt} = 0.08P(1 - \frac{P}{1000})$$

1. For what values of P is the population increasing/decreasing?

increasing: $\frac{dP}{dt} > 0$, $0 < P < 1000$

decreasing: $\frac{dP}{dt} < 0$ $P < 0$ OR $P > 1000$

2. What are the equilibrium solutions?

$\frac{dP}{dt} = 0$; $P = 0$ OR $P = 1000$

3. It can be showed that the function

$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

is a solution of the differential equation above. Show that the population size will approach its carrying capacity for $t \to +\infty$.

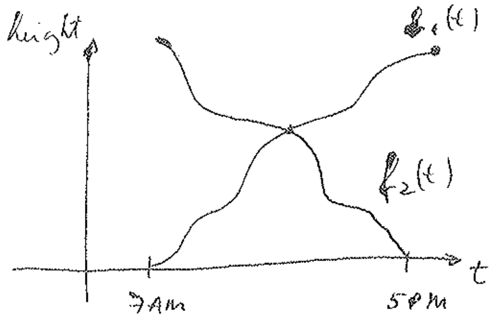
At $t \to \infty$, $e^{-0.08t} \longrightarrow 0$, so

$$P(t) \longrightarrow \frac{1000}{1+0} = 1000.$$

This is the carrying capacity.

Other practice problems suitable for the exam.

PROBLEM 1. [6 pts.] A hiker sets off on a mountain ascent at 7:00 a.m. in the morning. She reaches the summit at 5:00 p.m. The next morning, she begins her descent at 7:00 a.m., taking the same path back. She reaches the bottom of the mountain at 5:00 p.m. Show that there is a point on the path that the hiker will cross at exactly the same time of day on both days.



$f_1(t), f_2(t)$ - height at time t day 1, day 2.

check $f_1(t) - f_2(t) = 0$ for some t.

PROBLEM 2. Use derivation rules.

(1)[5 pts.] Find y' if $y = \cos(e^{\sqrt{\tan 3x}})$.

(2)[5 pts.] Let $f(x) = e^{mx}\cos nx$. Find f'.

(1) $y' = -\sin(e^{\sqrt{\ }}) \cdot e^{\sqrt{\ }} \cdot \frac{1}{2\sqrt{\ }} \cdot \sec^2 3x \cdot 3$

(2) $f'(x) = me^{mx} \cdot \cos nx + e^{mx}(-n) \cdot \sin(nx)$

PROBLEM 3. [10 pts.] The age of a painting can be determined by carbon dating. Fresh paint contains a certain amount of C-14 that decays exponentially. If we know the decay rate, we can determine the age by measuring the amount of C-14 in a painting.
A portrait of Shakespeare was measured to contain 97% of the original amount of C-14. It was determined to be a fake because a 400 year old painting should contain only 95% of the original amount of C-14. Find how long time ago this picture was painted. Leave logarithms in your final answer.

$f(t)$ = amount of C-14 at time t in years

$f(0) = 1$ (100% at time 0).

$f(t) = \cancel{f(0)}e^{kt} = e^{kt}$

Find k:

$f(400) = .95$

$e^{400k} = .95$

$400k = \ln(.95)$

$k = \frac{\ln(.95)}{400}$

Find t:

$f(t) = .97$

$e^{kt} = .97$

$kt = \ln(.97)$

$t = \frac{\ln(.97)}{k} = 400 \cdot \frac{\ln(.97)}{\ln(.95)}$