SCIENCE ONE – MATHEMATICS
MIDTERM EXAM
OCTOBER 22, 2014

Name: SOLUTIONS
Student ID:

Exam rules:

- No calculators, other electronic devices, open books or notes are allowed.
- There are 7 problems in this exam. Use opposite empty pages if needed.

1		10
2		10
3		10
4		10
5		8
6		6
7		6
Total		60

PROBLEM 1. Use limit rules to find the following limits. If the limit does not exist, explain why. No $\varepsilon - \delta$ proofs are required.

1.[5 pts.]

$$\lim_{x \to -2} \frac{1}{x+2} + \frac{4}{x^2-4}.$$

$$\frac{1}{x+2} + \frac{4}{(x-2)(x+2)} = \frac{x-2+4}{(x-2)(x+2)}$$

$$= \frac{x+2}{(x-2)(x+2)} = \frac{1}{x-2} \xrightarrow[x \to -2]{} \frac{1}{-4}$$

2.[5 pts.]

$$\lim_{x \to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}.$$

$$\frac{\sqrt{x+6}-x}{x^3-3x^2} \cdot \frac{(\sqrt{x+6}+x)}{(\sqrt{x+6}+x)}$$

$$= \frac{x+6-x^2}{(x^3-3x^2)(\sqrt{x+6}+x)} = \frac{-(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6}+x)}$$

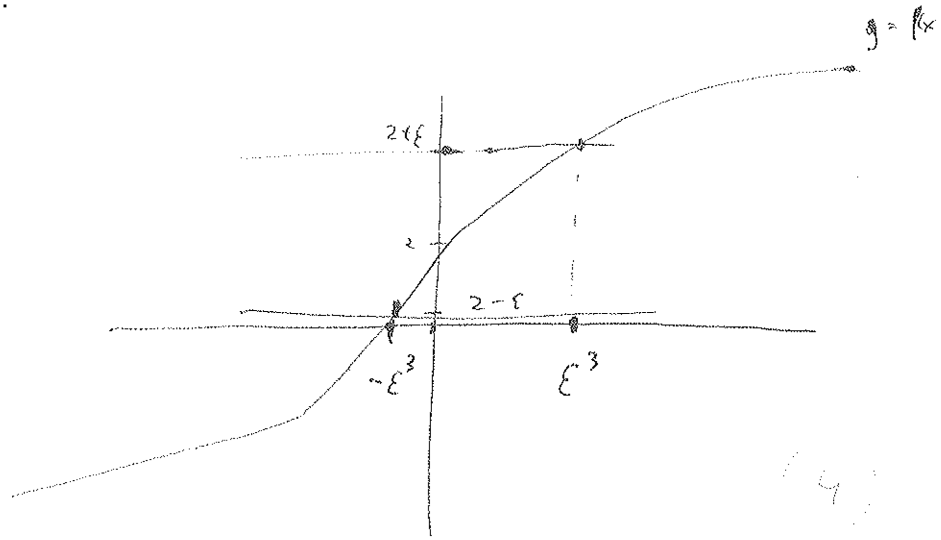
$$= \frac{-(x+2)}{x^2(\sqrt{x+6}+x)} \xrightarrow[x \to 3]{} \frac{-5}{9 \cdot (\sqrt{9}+3)} = \frac{-5}{54}$$

PROBLEM 2. [10 pts.] Use the $\varepsilon - \delta$ definition of limits to prove that

$$f(x) = 2 + \sqrt[3]{x}$$

is continuous at $x = 0$.

1. Find δ:



(4)

2. We need to prove

$$\lim_{x \to 0} f(x) = f(0) = 2.$$

For any $\varepsilon > 0$, take $\delta = \varepsilon^3$.

We need to show:

$$\text{if } |x - 0| < \delta \text{ then } |f(x) - 2| < \varepsilon.$$

Simplifying, this means:

$$\text{if } |x| < \varepsilon^3 \text{ then } \sqrt[3]{|x|} < \varepsilon.$$

(6)

We can take the cube root of both sides in $|x| < \varepsilon^3$ to obtain $\sqrt[3]{|x|} < \varepsilon$.

PROBLEM 3. Use derivation rules.

1.[5 pts.] Find the derivatives of $h(x)$, $g(x)$ and $f(x)$ at $x = 1$, where

$$h(x) = e^{x^2}, \quad g(x) = \sin(\pi x), \quad f(x) = h(g(x^2) + x).$$

$h'(x) = e^{x^2} \cdot 2x \qquad h'(1) = e \cdot 2$

$g'(x) = \cos(\pi x) \cdot \pi \qquad g'(1) = \cos(\pi) \cdot \pi = -\pi$

$f'(x) = h'(\underbrace{g(x^2) + x}_{1}) \cdot [\, g'(\underset{1}{\underset{\parallel}{x^2}}) \cdot \underbrace{2x}_{2} + 1 \,]$

$= h'(1) \cdot [\, g'(1) \cdot 2 + 1 \,]$

$= e \cdot 2 \cdot [-2\pi + 1]$

$= 2e\,(1 - 2\pi)$

2.[5 pts.] Find the slope of the tangent line to the curve defined by the equation

$$e^{x^2 y} = x + 2y$$

at the point $(1, 0)$.

Differentiate both sides:

$\underbrace{e^{x^2 y}}_{1} \cdot (\underset{0}{\underset{\parallel}{2xy}} + \underbrace{x^2}_{1} y') = 1 + 2y'$

$y' = 1 + 2y'$

$y' = -1$

PROBLEM 4. [10 pts.] Let

$$f(x) = \begin{cases} 4x - 1 - \frac{\sin^2(2x-2)}{3x-3} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1. \end{cases}$$

Use the definition of derivative as a limit to find $f'(1)$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim \frac{4(1+h) - 1 - \frac{\sin^2(2(1+h)-2)}{3(1+h)-3} - 3}{h}$$

$$= \lim \frac{4h - \frac{\sin^2(2h)}{h}}{h}$$

$$= \lim_{h \to 0} 4 - \frac{\sin(2h)}{h} \cdot \frac{\sin(2h)}{h}$$

$$= 4 - 2 \cdot 2 = 0$$

PROBLEM 5. [8 pts.] Find all values a, b, such that the function

$$f(x) = \begin{cases} 6x - 2 & \text{if } x < a \\ x^2 + bx + 2 & \text{if } x \geq a \end{cases}$$

is continuous and differentiable everywhere.

Same value at a:

$$6a - 2 = a^2 + ba + 2$$

Same slope at a:

$$6 = 2a + b$$

Solve for a, b:

$$b = 6 - 2a$$

$$6a - 2 = a^2 + (6 - 2a)a + 2$$

$$a^2 = 4, \quad a = \pm 2 \qquad b = 6 \pm 4$$

$$\begin{cases} a = -2 \\ b = 10 \end{cases} \quad \text{OR} \quad \begin{cases} a = 2 \\ b = 2 \end{cases}$$

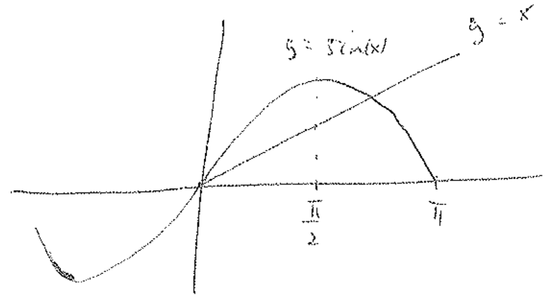
PROBLEM 6. [6 pts.] Show that there is at least one point on the curve

$$y = 4\cos x + x^2,$$

with $x > 0$, where the tangent line is horizontal.

$y' = -4\sin x + 2x = 0$

$2\sin x - x = 0$



(2) $f(x) = 2\sin x - x$

$f\left(\frac{\pi}{2}\right) = 2 - \frac{\pi}{2} = 2 - 1.5.. > 0$

(3) $f(\pi) = 0 - \pi = -\pi < 0$

(1) $\Rightarrow f(x) = 0$ for some x in $\left(\frac{\pi}{2}, \pi\right)$.

PROBLEM 7. [6 pts.] Consider the logistic equation

$$\frac{dP}{dt} = 0.04P(1500 - P)$$

1. Find all equilibrium solutions.

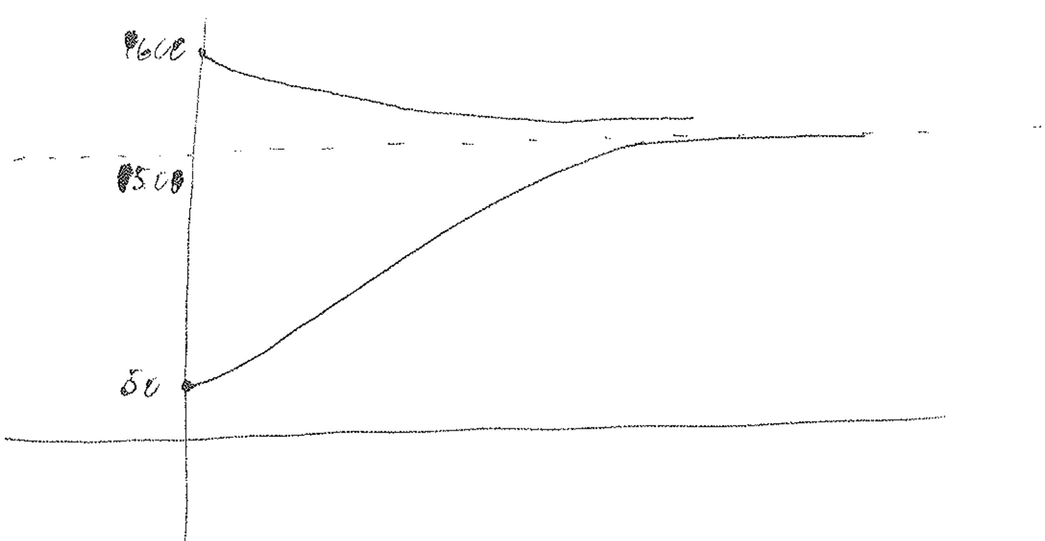
$dP = 0 \implies P = 0$ OR $P = 1500$

2. Describe the short-term and long-term behaviour of a solution $P(t)$ with initial value $P(0) = 50$. (Does $P(t)$ increase or decrease? What happens to $P(t)$ as $t \to \infty$?)

$\frac{dP}{dt}(50) > 0$. $P(t)$ increases, approaches 1500 as $t \to \infty$

3. Describe how the behaviour of a solution with initial value $P(0) = 1600$ differs from the above case. (Does $P(t)$ increase/decrease, what is the limit?)

$P(t)$ decreases, approaches 1500 as $t \to \infty$



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