Science One, Mathematics, HW #6 Solutions

1.(a) We can approximate the partial sum

$$S_n \approx \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1).$$

Let's now find n = number of seconds in 13 billion years. Assume a year has 364.25 days.

$$n = 13 \cdot 10^9 \cdot 364.25 \cdot 24 \cdot 60 \cdot 60 \approx 4 \cdot 10^{17}$$

Now take

$$\ln(n+1) \approx 40.5$$

(b). We need to find N, such that

$$S_N = 2 \cdot S_n ,$$

$$\ln(N+1) = 2 \ln(n+1) = (n+1)^2$$

$$N+1 = (n+1)^2$$

Thus, it takes approximately $n = 4 \cdot 10^{17}$ times the current age of the universe until the sum doubles to $2 \cdot 40.5 = 91$

2. (a) Let E_n be the number of edges in the snowflake at step n:

$$E_0 = 3$$
$$E_1 = 4 \cdot 3$$
$$E_2 = 4^2 \cdot 3$$
$$\dots$$
$$E_n = 4^n \cdot 3$$

Note: at every step we replace each edge with 4 edges.

Let L_n be the length of an edge at step n.

Then $L_n = \frac{1}{3} \cdot L_{n-1} = \dots = \frac{1}{3^n} \cdot L_0$

Now the perimeter at step n is

$$E_n \cdot L_n = 4^n \cdot 3 \cdot \frac{1}{3^n} \cdot L_0 = \left(\frac{4}{3}\right)^n \cdot 3 L_0$$
$$= \left(\frac{4}{3}\right)^n \cdot (\text{Perimeter at step } 0)$$

(b) A each step n, we add E_{n-1} new triangles (one on each edge of the previous step). Each of these triangles has side length L_n, hence area $\left(\frac{L_n}{L_0}\right)^2 \times$ area of triangle at time 0.

Thus, the area after step n is:

$$A_0 + E_0 \cdot \left(\frac{L_1}{L_0}\right)^2 \cdot A_0 + E_1 \left(\frac{L_2}{L_0}\right)^2 A_0 + \dots + E_{n-1} \left(\frac{L_n}{L_0}\right)^2 A_0$$
$$= A_0 + 3 \cdot \left(\frac{1}{3}\right)^2 \cdot A_0 + 4 \cdot 3 \left(\frac{1}{3^2}\right)^2 A_0 + \dots + 4^{n-1} \cdot 3 \left(\frac{1}{3^n}\right)^2 A_0$$
$$= A_0 + \sum_{k=1}^{n} 4^{k-1} \cdot 3 \cdot \frac{1}{3^{2k}} A_0$$
$$= A_0 + \sum_{k=0}^{n-1} 4^k \cdot \frac{1}{3^{2k}} \cdot \frac{1}{3} A_0 = A_0 + \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}} \cdot \frac{1}{3} A_0$$
$$\xrightarrow{n \to \infty} A_0 + \frac{1}{3 - \frac{4}{3}} A_0 = A_0 + \frac{3}{5} A_0 = \frac{8}{5} A_0$$

3. (a) We need to prove:

$\prod (1+a_n)$ converges $\iff$ $\sum \ln(1+a_n)$ converges $\iff$ $\sum a_n$ converges.

The first implication is a definition (of when the product converges), it does not need a proof. We prove the second implication using limit comparison theorem:

If $a_n \underset{n\to\infty}{\not\to} 0$ then both series are infinite. So, we may assume that $\lim_{n\to\infty} a_n = 0$. Let's compute

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{\text{L'Hop.}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

By limit comparison theorem

$$\sum \ln(1+a_n) \text{ converges } \iff \sum a_n \text{ converges.}$$

(b) Write the product as

$$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}} = \prod_{n=2}^{\infty} \Big(1 + \underbrace{\left(\frac{1}{1-n^{-s}} - 1\right)}_{a_n}\Big)$$

Let's show that the series $\sum a_n$ converges. Then the product converges by the theorem.

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n^{-s}}{1-n^{-s}} = \sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

We compare this series to $\sum \frac{1}{n^s}$ using limit comparison theorem:

$$\lim_{n\to\infty} \frac{\frac{1}{n^s-1}}{\frac{1}{n^s}} = \lim_{n\to\infty} \frac{n^s}{n^s-1} = \lim_{n\to\infty} \frac{1}{1-n^{-s}} = 1.$$

Since we know $\sum \frac{1}{n^s}$ converges $(s>1)$, the limit comparison theorem implies that $\sum \frac{1}{n^s-1}$ also converges, hence the product converges.