SCIENCE ONE, MATHEMATICS - HOMEWORK #6

Due 10AM, Tuesday, March 17

PROBLEM 1. The harmonic series is the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Even though we know the sum to be infinite, it grows extremely slowly. It is said that the harmonic series is proved to diverge but not observed to do so.

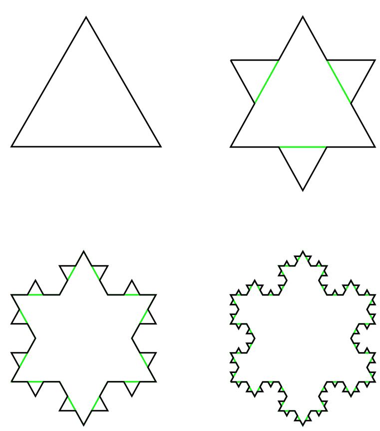
(a) Suppose we start with the first term $\frac{1}{1}$ at the time the universe was formed, 13 billion years ago, and add a new term every second. How big would the sum be today? (Think of the partial sum s_n as the left endpoint Riemann sum of the integral

$$\int_1^{n+1} \frac{1}{x} dx$$

and use this integral to approximate the partial sum.)

(b) How long do we need to wait (in terms of multiples of 13 billion years) for the sum to double.

PROBLEM 2. The Koch snowflake (named after Helge von Koch) is constructed step by step as follows. Start with the equilateral triangle at step zero. At step one modify each side of the triangle by adding to it an equilateral triangle one third of the size of the original triangle. Repeat this process. The Koch snowflake is the limit of this process.



(a) Find the length of the perimeter after step n (in terms of the perimeter of the original triangle). Show that the length approaches infinity.

(b) Find the area of the snowflake after step n. Find the limit of this area.

Problem 3. We say that an infinite product

$$\prod_{n=0}^{\infty}(1+a_n) = (1+a_0)(1+a_1)(1+a_2)\cdots$$

converges to a limit P if (after applying ln to both expressions), the series

$$\sum_{n=0}^{\infty} \ln(1+a_n)$$

converges to $\ln P$.

Theorem 0.1. *Let a_i be non-negative numbers. Then the infinite product above converges if and only if the series*

$$\sum_{n=0}^{\infty} a_n$$

converges.

(a) Prove the theorem. (Hint: Use the limit comparison test to compare the two series. You may assume that $a_n \to 0$ as $n \to \infty$.)

(b) Prove that the infinite product

$$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}}$$

converges for $s > 1$.