Science One - Mathematics Homework #5.

Problem 1.

(a) <u>Base</u> $n=1$.
<u>check</u> $1^3 = \left[\frac{1(1+1)}{2}\right]^2$ ✓

<u>Step</u>. Assume we know for n:

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Compute the sum for $n+1$:

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2 + 2(n+1)^3}{4} = \frac{(n+1)^2\left[n^2 + 4(n+1)\right]}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4} = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

(b) <u>Base</u> $n=1$:

$$1^5 = \frac{1^2(1+1)^2(2+2-1)}{12}$$

<u>Step</u> Assume we know for n. Prove for $n+1$:

$$\sum_{i=1}^{n+1} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} + (n+1)^5$$

$$= \frac{(n+1)^2\left[n^2(2n^2+2n-1) + 12(n+1)^3\right]}{12}$$

Need to show:

$$n^2(2n^2+2n-1) + 12(n+1)^3 \stackrel{?}{=} (n+2)^2\left(2(n+1)^2 + 2(n+1) - 1\right)$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$2n^4 + 2n^3 - n^2 + 12n^3 + 36n^2 + 36n + 12 \quad\Big|\quad (n^2+4n+4)(2n^2+6n+3)$$

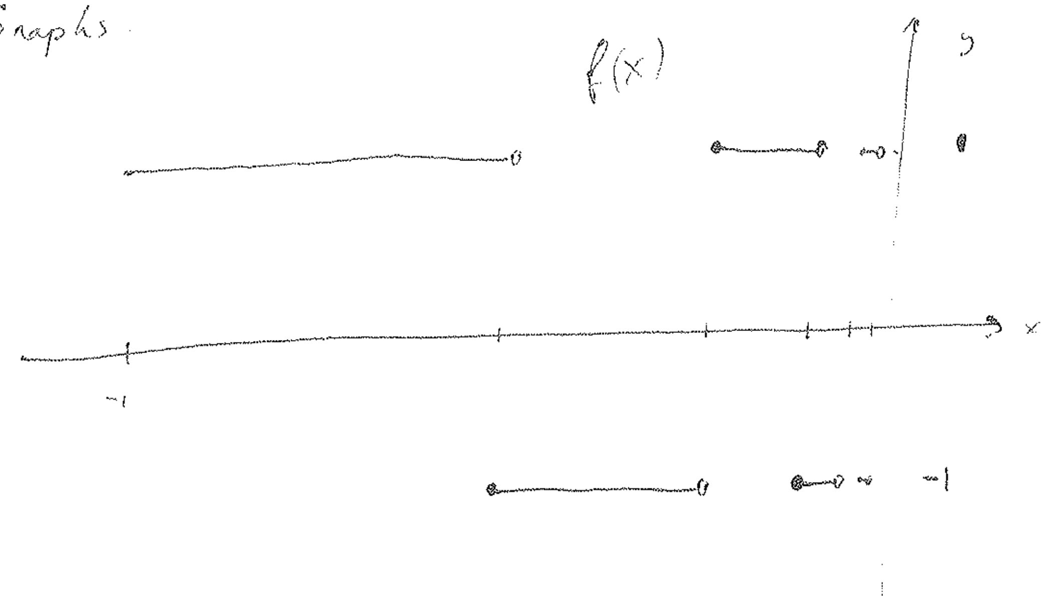
$$\parallel$$

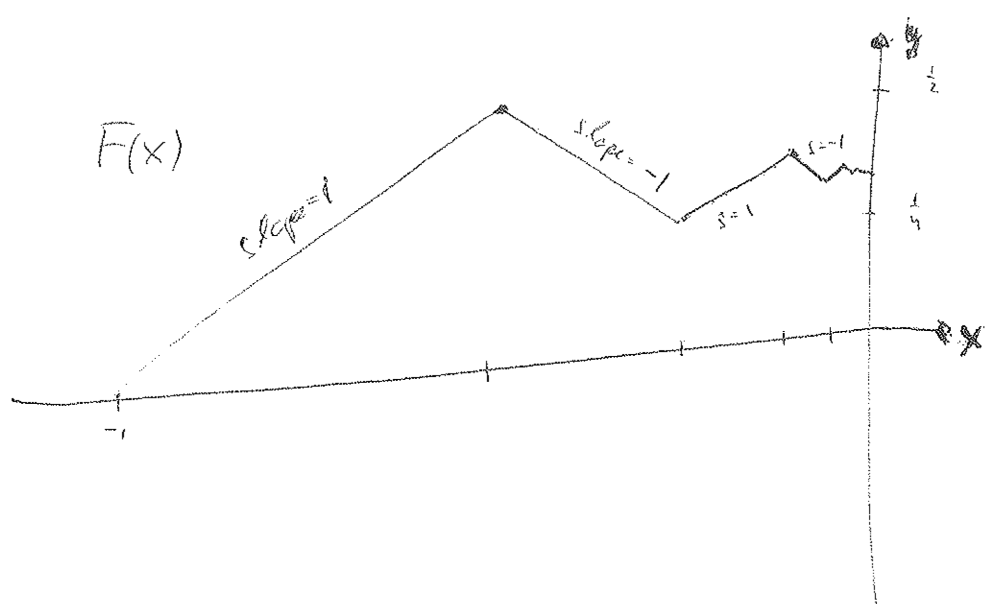
$$2n^4 + 6n^3 + 3n^2 + 8n^3 + 24n^2 + 12n + 8n^2 + 24n + 12$$

$$\parallel$$

$$2n^4 + 14n^3 + 35n^2 + 36n + 12$$

2. Graphs





$$\lim_{x \to 0} F(x) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots = -\sum_{i=1}^{\infty} \left(-\frac{1}{2}\right)^i = L$$

To find L:

$$L = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

$$+ \quad \frac{1}{2} L = \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

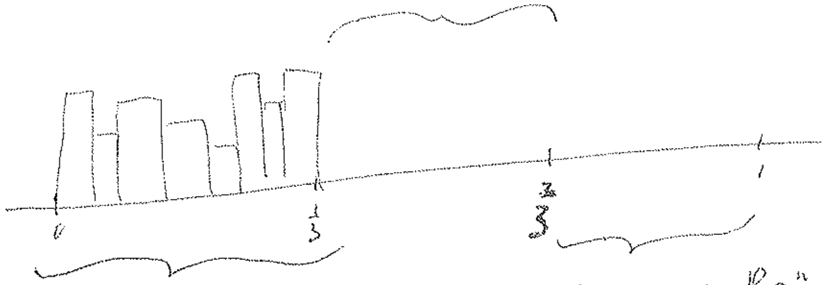
$$\frac{3}{2} L = \frac{1}{2}, \qquad L = \frac{1}{3}$$

3. (a) Prove by induction that $R_{3^n} = 0$ for all $n \geq 0$.

<u>n=0</u> $R_1 = \Delta x \cdot f(.5) = 1 \cdot 0 = 0$, $.5 \notin C$.

<u>step.</u> Assume $R_{3^n} = 0$.
Compute $R_{3^{n+1}}$:

All rectangles have height 0.



3^n rectangles, same as for R_{3^n}, squeezed 3 times. All have height 0.

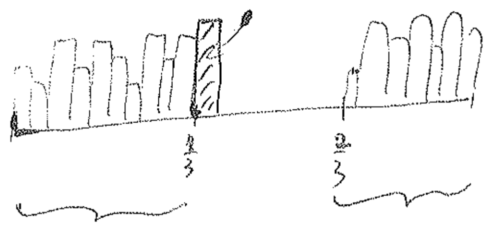
Same as R_{3^n} squeezed by factor of $\frac{1}{3}$

$$R_{3^{n+1}} = \frac{1}{3} R_{3^n} + \frac{1}{3} R_{3^n} = \frac{2}{3} R_{3^n} = \frac{2}{3} \cdot 0 = 0.$$

(b) Formula:

$$R_{3^{n+1}} = \frac{2}{3} R_{3^n} + \frac{1}{3^{n+1}}$$

extra rectangle.



$\frac{1}{3} \cdot R_{3^n}$ $\qquad$ $\frac{1}{3} R_{3^n}$

n	R_{3^n}	
0	1	← compute R_1 directly
1	1	use formula to compute
2	$\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$	
3	$\frac{2}{3} \cdot \frac{7}{9} + \frac{1}{27} = \frac{15}{27}$	
4	$\frac{2}{3} \cdot \frac{15}{27} + \frac{1}{3 \cdot 27}$	

$$= \frac{31}{3^4} = \frac{32-1}{3^4} = \frac{2^5 - 1}{3^4}$$

Hypothesis: $R_{3^n} = \dfrac{2^{n+1} - 1}{3^n}$

<u>Proof by induction:</u>

<u>n=0</u> $R_1 = \frac{2-1}{1} = 1$ ✓

<u>step.</u> Use formula:

$$R_{3^{n+1}} = \frac{2}{3} \frac{2^{n+1} - 1}{3^n} + \frac{1}{3^{n+1}} = \frac{2^{n+2} - 1}{3^{n+1}}$$ ✓

SCIENCE ONE, MATHEMATICS - HOMEWORK #5

Due 10AM, Monday, Jan. 26

PROBLEM 1. Use induction to prove

(a)

$$\sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2.$$

(b)

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

PROBLEM 2. This problem studies integrals of functions such as $\sin(1/x)$. We replace the sin function with a simpler function $f(x)$ that similarly oscillates between -1 and 1. The function $f(x)$ is defined on the interval $[-1,0)$ as follows. First divide $[-1,0)$ into intervals of length $1/2, 1/4, 1/8, 1/16, \ldots$, namely $[-1,-1/2), [-1/2,-1/4), [-1/4,-1/8), \ldots$. Then let $f(x)$ take the value $+1$ on the first interval, -1 on the second interval, $+1$ on the third interval, and so on, alternating between $+1$ and -1. In short,

$$f(x) = \begin{cases} +1 & \text{if } x \in [-1, -\frac{1}{2}) \cup [-\frac{1}{4}, -\frac{1}{8}) \cup \ldots \\ -1 & \text{if } x \in [-\frac{1}{2}, -\frac{1}{4}) \cup [-\frac{1}{8}, -\frac{1}{16}) \cup \ldots \end{cases}$$

Sketch the graph of

$$F(x) = \int_{-1}^{x} f(t)dt.$$

In particular, find $\lim_{x\to 0} F(x)$. (You may assume here that $f(x)$ is integrable and find the integral by computing areas under the graph. No Riemann sums are needed.)

PROBLEM 3. The Cantor set C is a subset of the interval $[0,1]$ constructed as follows. Start with $[0,1]$. Remove the middle third, $(1/3, 2/3)$. Then remove the middle third of each of the two remaining intervals. Continue this way, removing the middle third of each remaining interval from the previous step. The Cantor set is a fractal. It has a self-similarity property: the first third of it, lying in $[0,1/3]$, is the same as the whole Cantor set, only squeezed by a factor $1/3$. Similarly, the last third is the whole Cantor set squeezed by $1/3$. Let $f(x)$ be the function on $[0,1]$ defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

(a)Find the Riemann sum R_{3^n} for

$$\int_0^1 f(x)dx$$

using the midpoint rule, where $n \geq 0$ is any integer.