SCIENCE ONE, MATHEMATICS - HOMEWORK #5

Due 10AM, Monday, Jan. 26

PROBLEM 1. Use induction to prove

(a)

$$\sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2.$$

(b)

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

PROBLEM 2. This problem studies integrals of functions such as $\sin(1/x)$. We replace the sin function with a simpler function $f(x)$ that similarly oscillates between -1 and 1. The function $f(x)$ is defined on the interval $[-1, 0)$ as follows. First divide $[-1, 0)$ into intervals of length $1/2, 1/4, 1/8, 1/16, \ldots$, namely $[-1, -1/2), [-1/2, -1/4), [-1/4, -1/8), \ldots$. Then let $f(x)$ take the value $+1$ on the first interval, -1 on the second interval, $+1$ on the third interval, and so on, alternating between $+1$ and -1. In short,

$$f(x) = \begin{cases} +1 & \text{if } x \in [-1, -\frac{1}{2}) \cup [-\frac{1}{4}, -\frac{1}{8}) \cup \ldots \\ -1 & \text{if } x \in [-\frac{1}{2}, -\frac{1}{4}) \cup [-\frac{1}{8}, -\frac{1}{16}) \cup \ldots \end{cases}$$

Sketch the graph of

$$F(x) = \int_{-1}^{x} f(t)dt.$$

In particular, find $\lim_{x \to 0} F(x)$. (You may assume here that $f(x)$ is integrable and find the integral by computing areas under the graph. No Riemann sums are needed.)

PROBLEM 3. The Cantor set C is a subset of the interval $[0, 1]$ constructed as follows. Start with $[0, 1]$. Remove the middle third, $(1/3, 2/3)$. Then remove the middle third of each of the two remaining intervals. Continue this way, removing the middle third of each remaining interval from the previous step. The Cantor set is a fractal. It has a self-similarity property: the first third of it, lying in $[0, 1/3]$, is the same as the whole Cantor set, only squeezed by a factor 1/3. Similarly, the last third is the whole Cantor set squeezed by 1/3. Let $f(x)$ be the function on $[0, 1]$ defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

(a)Find the Riemann sum R_{3^n} for

$$\int_0^1 f(x)dx$$

using the midpoint rule, where $n \geq 0$ is any integer.

(b) Find the Riemann sum R_{3^n} using the left endpoint rule. (Hint: use the self-similarity property. For example, when computing R_{27}, then the 9 rectangles on the first third of the interval look exactly like the rectangles that compute R_9, except that their width is squeezed by a factor $1/3$. Using this, one can express R_{3^n} in terms of $R_{3^{n-1}}$. Don't forget the one nonzero rectangle lying in the middle third.)

More about the Cantor set. The description of the Cantor set by repeatedly removing the middle third of each remaining interval does not tell you which points are left in C. It is not hard to convince yourself that after doing a few steps of the construction we are left with a finite set of intervals, and the endpoints of these intervals will not be removed in the following steps (because we only remove intervals in the middle). Thus, after the first step of the construction we have removed the middle third $(1/3, 2/3)$ and are left with two intervals $[1, 1/3]$ and $[2/3, 1]$. The endpoints of these intervals $1, 1/3, 2/3, 1$ are in the Cantor set. So, when we compute R_3 using the left endpoint rule, all rectangles will have height 1. Similarly, after two steps of the construction we are left with four intervals $[0, 1/9]$, $[2/9, 3/9]$, $[6/9, 7/9]$, $[8/9, 9/9]$. The endpoints of these intervals, $0, 1/9, 2/9, 3/9, 6/9, 7/9, 8/9, 1$ are in C. Computing the Riemann sum R_9 using the left endpoint rule, we see that 7 rectangles have height 1 and two have height 0.