Q1. $P(r) = \frac{4r^2}{a^2} e^{-\frac{2}{a}r}$, $r \geq 0$

Domain: $r \geq 0$

Intercepts: $P(0) = 0$ so $(0,0)$

Limits: $\lim_{r\to\infty} \frac{4r^2}{a^2} e^{-\frac{2}{a}r} \underset{[\infty \cdot 0]}{=} \lim_{r\to\infty} \frac{4r^2}{a^2 e^{\frac{2}{a}r}} = [\text{l'Hopital's rule}]$

$= \lim_{r\to\infty} \frac{4 \cdot 2r}{a^2 \cdot \frac{2}{a} e^{\frac{2}{a}r}} = \lim_{r\to\infty} \frac{8}{2\cancel{a} \frac{2}{\cancel{a}} e^{\frac{2}{a}r}} = 0$

$y = 0$ is H.A. when $r \to \infty$

Extrema:

$P'(r) = \frac{4}{a^2}\left(2re^{-\frac{2}{a}r} + r^2 e^{-\frac{2}{a}r} \cdot \left(-\frac{2}{a}\right)\right) = \frac{4}{a^2} \cdot 2re^{-\frac{2}{a}r}\left(1 - \frac{r}{a}\right)$

CP. $P'(r) = 0$ when $r = 0$ and $r = a$

$P' > 0$ when $r > 0$ always

$e^{-\frac{2}{a}r} > 0$ always

$1 - \frac{r}{a} > 0$ when $0 < r < a$

| | 0 | | a | |
|---|---|---|---|---|
| $P'$ | | + | | − |
| | | ↗ | Max | ↘ |

$P(a) = 4e^{-2}$ local max

Concavity:

$P' = \frac{8}{a^2} e^{-\frac{2}{a}r}\left(r - \frac{r^2}{a}\right)$

$P'' = \frac{8}{a^2}\left(e^{-\frac{2}{a}r} \cdot \left(-\frac{2}{a}\right) \cdot \left(r - \frac{r^2}{a}\right) + e^{-\frac{2}{a}r} \cdot \left(1 - \frac{2}{a}r\right)\right) =$

$$= \frac{8}{a^2} e^{-\frac{2}{a}r} \left( -\frac{2}{a}\left(r - \frac{r^2}{a}\right) + 1 - \frac{2}{a}r \right) =$$

$$= \frac{8}{a^2} e^{-\frac{2}{a}r} \left( -\frac{2}{a}r + \frac{2}{a^2}r^2 + 1 - \frac{2}{a}r \right) =$$

$$= \frac{8}{a^2} \frac{1}{a^2} e^{-\frac{2}{a}r} \left( 2r^2 - 4ar + a^2 \right)$$

$P''(r) = 0$

$$2r^2 - 4ar + a^2 = 0$$

$$r = \frac{4a \pm \sqrt{16a^2 - 4 \cdot 2 \cdot a^2}}{4} = \frac{4a \pm 2\sqrt{2}a}{4} = \left(1 \pm \frac{\sqrt{2}}{2}\right)a$$

$P'' > 0$ when $2r^2 - 4ar + a^2 > 0$

$$\left(r - \left(1 + \frac{\sqrt{2}}{2}\right)a\right)\left(r - \left(1 - \frac{\sqrt{2}}{2}\right)a\right) > 0$$

| | $0$ | | $\left(1-\frac{\sqrt{2}}{2}\right)a$ | | $\left(1+\frac{\sqrt{2}}{2}\right)a$ | |
|---|---|---|---|---|---|---|
| $P''$ | | $+$ | | $-$ | | $+$ |
| | | $\cup$ | | $\cap$ | | $\cup$ |
| | | | I.P. | | I.P. | |

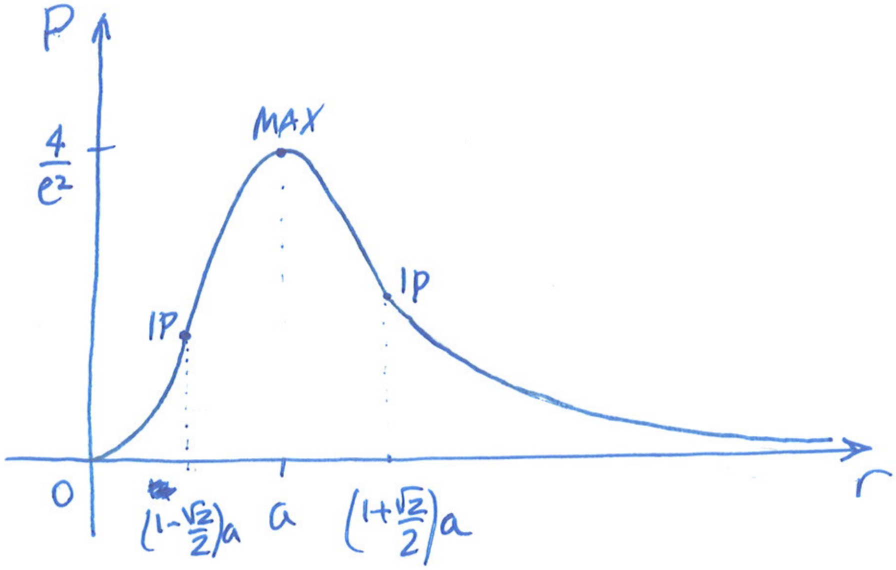
so $r < \left(1 - \frac{\sqrt{2}}{2}\right)a$ and $r > \left(1 + \frac{\sqrt{2}}{2}\right)a$

inflection pts $P\left(\left(1 - \frac{\sqrt{2}}{2}\right)a\right) = \frac{4}{\cancel{a^2}}\left(1 - \frac{\sqrt{2}}{2}\right)^2 \cancel{a^2} e^{-\frac{2}{\cancel{a}}\left(1 - \frac{\sqrt{2}}{2}\right)\cancel{a}} =$

$$= 4\left(\frac{3}{2} - \sqrt{2}\right) e^{-2+\sqrt{2}}$$

$$P\left(\left(1 + \frac{\sqrt{2}}{2}\right)a\right) = \frac{4}{\cancel{a^2}}\left(1 + \frac{\sqrt{2}}{2}\right)^2 \cancel{a^2} e^{-\frac{2}{\cancel{a}}\left(1 + \frac{\sqrt{2}}{2}\right)\cancel{a}}$$

$$= 4\left(\frac{3}{2} + \sqrt{2}\right) e^{-2-\sqrt{2}}$$

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Q2

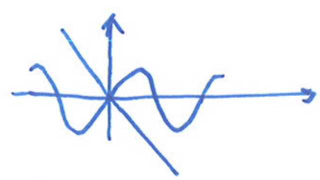
$y = x + \sin x$

Domain : $\forall x \in \mathbb{R}$

Interceps: $f(x) = 0$

$x + \sin x = 0$

$\sin x = -x$



only one solution at $x = 0$

So there is only one intercept at $(0, 0)$

Limits $\lim\limits_{x \to \pm\infty} (x + \sin x) = \pm\infty$

Periodicity : Since $\sin x$ is periodic with period $2\pi$ let's check if $f(x) = x + \sin x$ is also periodic

$f(x + 2\pi) = x + 2\pi + \sin(x + 2\pi) = x + 2\pi + \sin x$

$f(x + 2\pi) = f(x) + 2\pi$

the graph of f is shifted upwards

f is not periodic, however we can

restrict our study to the interval $[0, 2\pi]$, then extend our results to the entire domain using the fact that the graph is shifted vertically by $2\pi$ for every increment of $2\pi$ in $x$ to the right.

Extrema:

$f' = 1 + \cos x$

C.P. on $[0, 2\pi]$

$$1 + \cos x = 0$$
$$\cos x = -1$$
$$x = \pi$$

$$1 + \cos x > 0$$
$$\cos x > -1 \text{ for all } x$$

| | 0 | | $\pi$ | | $2\pi$ |
|---|---|---|---|---|---|
| $f'$ | | + | | + | |
| | | ↗ | | ↗ | |

on $[0, 2\pi]$ the function is always increasing there are no local max/min. In particular the slope of the tangent line at $x = \pi$ is 0 so the tangent line is horizontal.

Concavity $f'' = -\sin x$

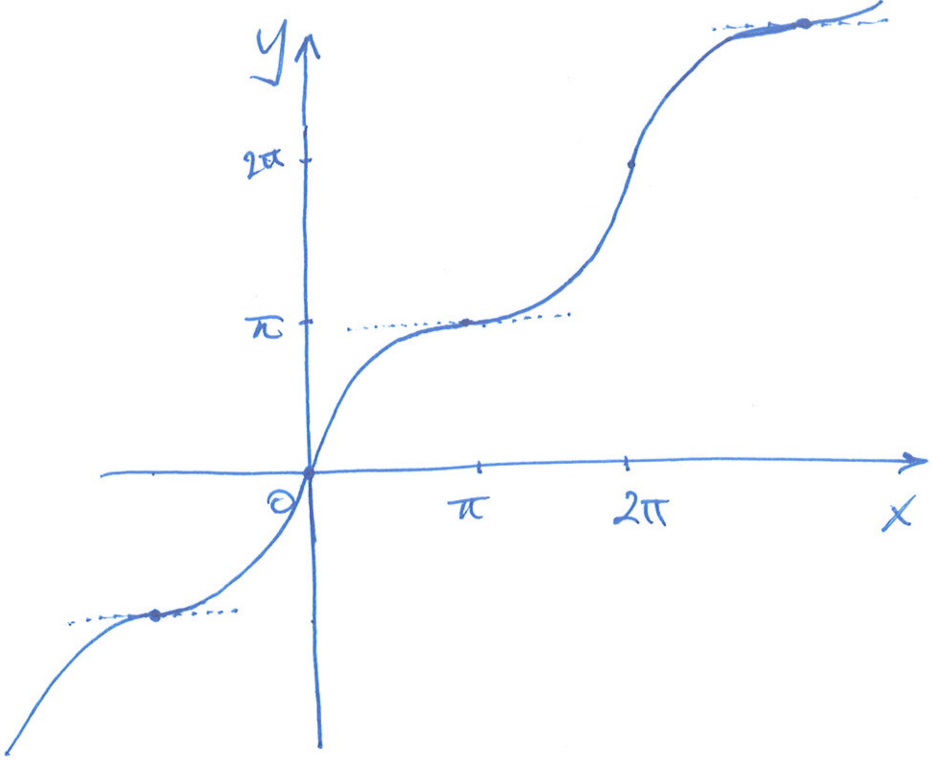
on $[0, 2\pi]$

$$-\sin x = 0$$
$$x = 0, \pi, 2\pi$$

$$-\sin x > 0$$
$$\sin x < 0$$
$$\pi < x < 2\pi$$

| | 0 | | $\pi$ | | $2\pi$ |
|---|---|---|---|---|---|
| $f''$ | | − | | + | |
| | | ∩ | IP | ∪ | |

$f(\pi) = \pi + \sin \pi = \pi$ inflection point


y
2π
π
0
π
2π
x

3. Taylor series for $\ln(x)$ at $a=1$.

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = (-1)\cdot x^{-2}$

$f'''(x) = (-1)(-2)\, x^{-3}$

$\dots$

$f^{(n)}(x) = (-1)^{n-1}\cdot(n-1)!\; x^{-n}$ $\qquad f^{(n)}(1) = (-1)^{n-1}\cdot(n-1)!$

$$T_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

(a) $T_2(x) = (x-1) - \frac{(x-1)^2}{2}$

$$R_2(x) = \frac{f'''(c)}{3!}(x-1)^3 \qquad 1 < c < x$$

$$= \frac{(x-1)^3}{3\cdot c^3}$$

We don't know $c$, but the $c$ that gives the biggest error is $c=1$ (if $c>1$, then $R_2(x)$ becomes smaller.):

$$|R_2(x)| < \frac{(x-1)^3}{3}$$

If $\frac{(x-1)^3}{3} < 0.005$, then $|R_2(x)| < 0.005$

$\Updownarrow$

$(x-1)^3 < 0.015$

$(x-1) < \sqrt[3]{0.015} = \frac{\sqrt[3]{15}}{10} \approx .25.$

Thus, if $x \in [1, 1.25]$, then we get 2 digit accuracy.

(b) $$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}\right| = \underbrace{\left|\frac{c^{-(n+1)}}{n+1}\cdot(x-1)^{n+1}\right|}_{\text{maximal when } x=2,\ c=1} \leq \frac{1}{n+1}$$

Thus, $|R_n(x)| < 0.005$ if

$$\frac{1}{n+1} < 0.005 \iff n+1 > \frac{1}{0.005} = 200$$

$n = 200$ works.

4. Find Taylor series: (at $a=0$)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\tan(x) = x + \frac{2}{3!}x^3 + \dots$$

$$\tan'(x) = \sec^2(x) = \frac{1}{\cos^2(x)} \qquad \tan'(0) = 1$$

$$\tan''(x) = \frac{2\sin(x)}{\cos^3(x)} \qquad \tan''(0) = 0$$

$$\tan'''(x) = \frac{2\cos^2(x) + 6\sin^2 x}{\cos^4(x)} \qquad \tan'''(0) = 2$$

$$\frac{1}{\ln(1+x)} - \frac{1}{\tan x} = \frac{1}{x - \frac{x^2}{2} + \dots} - \frac{1}{x + \frac{x^3}{3} + \dots}$$

$$= \frac{x + \frac{x^3}{3} + \dots - x + \frac{x^2}{2} + \dots}{\left(x - \frac{x^2}{2} + \dots\right)\left(x + \frac{x^3}{3} + \dots\right)}$$

$$= \frac{\frac{x^2}{2} + \frac{x^3}{3} + \dots}{x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \dots}$$

$$= \frac{x^2\left(\frac{1}{2} + \frac{x}{3} + \dots\right)}{x^2\left(1 - \frac{x}{2} + \dots\right)}$$

$$= \frac{\frac{1}{2} + \frac{x}{3} + \dots}{1 - \frac{x}{2} + \dots} \xrightarrow[x \to 0]{} \frac{1}{2}$$