# Science 1, Mathematics, Homework #3

1. (a) Write

$$y = \cot^{-1}(x)$$

$$\cot(y) = x$$

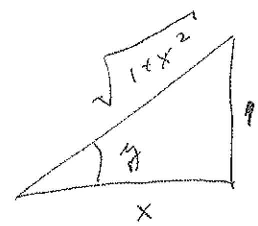
Now differentiate:

$$\frac{d}{dx}\left(\frac{\cos(y)}{\sin(y)}\right) = \frac{d}{dx}\, x\,,$$

$$\frac{-\sin^2(y) - \cos^2 y}{\sin^2(y)} \cdot y' = 1\,,$$

$$y' = -\sin^2(y)$$

$$y' = -\frac{1}{1+x^2}\,.$$




$$\cot(y) = \frac{x}{1} = x$$

$$\sin(y) = \frac{1}{\sqrt{1+x^2}}$$

(b) Derivative is:

$$\frac{d}{dx}\left(\tan^{-1}(x) + \cot^{-1}(x)\right) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0\,,$$

hence $\tan^{-1}(x) + \cot^{-1}(x) = C$ for some constant $C$.

Take $x = 0$, then

$$\tan^{-1}(0) + \cot^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}\,,$$

so $C = \frac{\pi}{2}$.

2. Let

$$f(x) = a_0 \cdot x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots + \frac{a_n}{n+1} x^{n+1}.$$

Then $f'(x) = p(x)$. Apply MVT to $f(x)$ in $[0,1]$. there exists $c$ in $(0,1)$, such that

$$f'(c) = p(c) = \frac{f(1) - f(0)}{1-0} = f(1) = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1}.$$

The right hand side is $0$ by assumption, so $c$ is a root of $p(x)$.

3. Apply MVT to $(1+x)^n$ in $(x,0)$, where $-1 \le x < 0$:

There exists $c$ in $(x,0)$, such that

$$n \cdot (1+c)^{n-1} = \frac{(1+0)^n - (1+x)^n}{0-x} = \frac{1-(1+x)^n}{-x}$$

Simplify this

$$(1+x)^n = 1 + n \cdot x \cdot (1+c)^{n-1}$$

Now

$$0 \le (1+c)^{n-1} < 1,$$

and because $x<0$, multiplying with $nx$:

$$0 \geq nx(1+c)^{n-1} > nx.$$

Hence

$$(1+x)^n > 1 + nx$$

when $-1 \le x < 0$.

4. (a) When $g(x) = x$, $g'(x) = 1$, then the GMVT is

$$[f(b) - f(a)] \cdot 1 = [b - a] \cdot f'(c)$$

for some $c$ in $(a, b)$. This is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let

$$h(x) = [g(b) - g(a)]\, f(x) - [f(b) - f(a)]\, g(x).$$

Applying the MVT to $h(x)$ in $[a, b]$, we get

$$h'(c) = \frac{h(b) - h(a)}{b - a} = 0$$

$$h'(c) = [g(b) - g(a)]\, f'(c) - [f(b) - f(a)]\, g'(c)$$

This gives the GMVT.

(c) Let $f(x)$, $g(x)$ be the position of Bolt and Gay, respectively, $x \in [0, 10]$. Applying the GMVT: there exists $c$ in $(0, 10)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{104.38}{102.99}$$

This is the same as

$$f'(c) = \frac{10.438}{10.299} \cdot g'(c).$$