SCIENCE ONE, MATHEMATICS - HOMEWORK #3

Due 10AM, Friday, Oct. 31

PROBLEM 1. In class we found the derivative of $\tan^{-1}(x)$. Let $\cot^{-1}(x)$ be the inverse function of $\cot(x) = \cos(x)/\sin(x)$, where we choose the branch $0 < \cot^{-1}(x) < \pi$.

(a)Find the derivative of $\cot^{-1}(x)$.

(b)Prove that

$$\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}.$$

(Hint: show that the derivative of the left hand side is zero.)

PROBLEM 2. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

then the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has a root x in the interval $(0,1)$. (Hint: apply the mean value theorem to a function with derivative $p(x)$.)

PROBLEM 3. The goal of this problem is to prove that for any integer $n > 0$ and any real number $x \geq -1$,

$$(1+x)^n \geq 1 + nx. \qquad (\diamond)$$

Using the binomial theorem, we can express

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + x^n = 1 + nx + (\ldots).$$

When $x \geq 0$, then the terms $(\ldots)$ are all non-negative, hence we have the inequality $(\diamond)$.
Prove the inequality $(\diamond)$ for $-1 \leq x < 0$ by applying the mean value theorem to the function $(1+x)^n$ in the interval $[x, 0]$.

PROBLEM 4. The generalized mean value theorem states that if $f(x), g(x)$ are continuous in $[a,b]$ and differentiable in (a,b), then there exists a number c in (a,b), such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

(a)Show that if $g(x) = x$, then the generalized mean value theorem reduced to the ordinary mean value theorem.

(b)Prove the generalized mean value theorem by applying the ordinary mean value theorem to the function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x).$$

(c)At the 2009 World Championships Usain Bolt ran 10 seconds with average velocity 10.438 m/s. At the same time Tyson Gay ran 10 seconds with average velocity 10.299 m/s. Prove that at some time t in $(0,10)$, the velocity of Usain Bolt was exactly 10.438/10.299 times the velocity of Tyson Gay.