① $g(x) = x^2 f(x)$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 f(x+h) - x^2 f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) f(x+h) - x^2 f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{x^2 (f(x+h) - f(x))}{h} + \lim_{h \to 0} \frac{\cancel{h}(2x+h) f(x+h)}{\cancel{h}}$$

$$= x^2 \cdot f'(x) + 2x \cdot f(x)$$

② $y = x^2$

The tangent at $P(t, t^2)$ has slope $2t$

The normal line at $P(t, t^2)$ has slope $-\frac{1}{2t}$, if $t \neq 0$.

Since the normal line goes through the point $Q(0, b)$, then

$$\frac{t^2 - b}{t - 0} = -\frac{1}{2t}$$

$$2t^2 - 2b = -1$$

$$t = \pm\sqrt{b - \tfrac{1}{2}}$$

If $b - \frac{1}{2} > 0$ (and $t \neq 0$), there are two distinct points

$P_1(\sqrt{b-\frac{1}{2}}, b - \frac{1}{2})$ and $P_2(-\sqrt{b-\frac{1}{2}}, b - \frac{1}{2})$

such that the normal line at each P goes through Q.

Thus there are two normal lines to the curve.

If $t = 0$, $P(0,0)$ and the tangent line at P is horizontal ($y=0$). Thus the normal line at P is vertical ($x=0$).

In conclusion, if $b - \frac{1}{2} > 0$, there are 3 normal lines to the curve that go through $Q(0, b)$.

If $b - \frac{1}{2} = 0$, there is only one point $P \equiv O(0,0)$ and thus only one normal line ($x=0$).

If $b - \frac{1}{2} < 0$ (and $t \neq 0$) the equation above has no solutions. The only normal line to the curve is found when $t = 0$, corresponding to the vertical line $x = 0$ through $O(0,0)$.

(3) $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x \qquad 0 < t < \infty$

$x(0) = x_0 \ , \quad x'(0) = v_0$

gen. sol. $\ x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

let $\ C_1 = A\cos\phi_0 \ , \quad C_2 = A\sin\phi_0$ where $0 < \phi_0 < 2\pi$

and $\ \omega = \sqrt{\frac{k}{m}}$

then $\ x(t) = A\cos\phi_0 \cos\omega t + A\sin\phi_0 \sin\omega t = A\cos(\omega t - \phi_0)$

$x'(t) = -\omega A \sin(\omega t - \phi_0)$

I.C. $\ x(0) = A\cos(-\phi_0) = A\cos(\phi_0) = x_0$

$x'(0) = -\omega A \sin(-\phi_0) = \omega A \sin\phi_0 = v_0$

We choose A and ϕ_0 so that

$A^2 = x_0^2 + \dfrac{v_0^2}{\omega^2} \qquad$ and $\qquad \cos\phi_0 = \dfrac{x_0}{A} \ , \quad \sin\phi_0 = \dfrac{v_0}{\omega A}$

$A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}} \qquad$ so that $\quad \tan\phi_0 = \dfrac{v_0}{\omega x_0}$

Note

$$\phi_0 = \begin{cases} \arctan\left(\dfrac{v_0}{\omega x_0}\right) & \text{when } \cos\phi_0 > 0,\ \sin\phi_0 > 0 \quad (\text{I quadrant}) \\ \pi + \arctan\left(\dfrac{v_0}{\omega x_0}\right) & \text{when } \cos\phi_0 < 0 \quad (\text{II or III quadrant}) \\ 2\pi + \arctan\left(\dfrac{v_0}{\omega x_0}\right) & \text{when } \cos\phi_0 > 0,\ \sin\phi_0 < 0 \quad (\text{IV quadrant}) \end{cases}$$

(a) $x_0 = 3$ m, $\quad v_0 = 1$ m/s

$A = \sqrt{9 + \dfrac{1}{\frac{40}{3}}} = \sqrt{9 + \dfrac{3}{40}} = \sqrt{\dfrac{363}{40}}$ m

$\cos\phi_0 > 0, \ \sin\phi_0 > 0 \Rightarrow \phi_0 = \arctan\left(\frac{1}{3}\sqrt{\frac{3}{40}}\right)$

$\tan\phi_0 = \frac{1}{3}\sqrt{\frac{3}{40}}$

(b) $x_0 = 3\ m$, $v_0 = -1\ m/s$

$$A = \sqrt{\frac{363}{40}}\ m$$

$\cos\phi_0 > 0$, $\sin\phi_0 < 0$

$$\tan\phi_0 = -\frac{1}{3}\sqrt{\frac{3}{40}} \Rightarrow \phi_0 = 2\pi + \arctan\left(\frac{v_0}{\omega x_0}\right)$$
$$= 2\pi + \arctan\left(-\frac{1}{3}\sqrt{\frac{3}{40}}\right)$$

(c) $x_0 = -3\ m$, $v_0 = 1\ m/s$

$$A = \sqrt{\frac{363}{40}}\ m$$

$\cos\phi_0 < 0$ $\sin\phi_0 > 0$

$$\tan\phi_0 = -\frac{1}{3}\sqrt{\frac{3}{40}} \Rightarrow \phi_0 = \pi + \arctan\left(\frac{v_0}{\omega x_0}\right)$$
$$= \pi + \arctan\left(-\frac{1}{3}\sqrt{\frac{3}{40}}\right)$$

(d) $x_0 = -3\ m$, $v_0 = -1\ m/s$

$$A = \sqrt{\frac{363}{40}}\ m$$

$\cos\phi_0 < 0$, $\sin\phi_0 < 0$

$$\tan\phi_0 = \frac{1}{3}\sqrt{\frac{3}{40}} \Rightarrow \phi_0 = \pi + \arctan\left(\frac{1}{3}\sqrt{\frac{3}{40}}\right)$$

④ $\dfrac{d^2u}{dx^2} = -\lambda u$, $0 < x < \pi$

$u(0) = 0$

$u'(\pi) = 0$

The general solution is $u(t) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$u(0) = C_1 \cos 0 + C_2 \sin 0 = 0$

$C_1 = 0$

$u'(x) = -C_1 \sin\sqrt{\lambda}x \cdot \sqrt{\lambda} + C_2 \cos\sqrt{\lambda}x \cdot \sqrt{\lambda}$

$u'(\pi) = C_2 \sqrt{\lambda} \cos\sqrt{\lambda}\pi = 0$

$C_2 \neq 0$ otherwise $u(x) \equiv 0$ trivial solution.

$\cos(\sqrt{\lambda}\pi) = 0$

$\sqrt{\lambda}\pi = \frac{\pi}{2} + k\pi$ $\quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$\sqrt{\lambda} = \frac{1}{2} + k$ $\quad k = 0, 1, 2, 3, \ldots$ we consider only the positive cases because of the symmetry of $\cos x$

$$\boxed{\lambda = \left(\frac{1}{2} + k\right)^2 \quad k = 0, 1, 2, 3 \ldots}$$