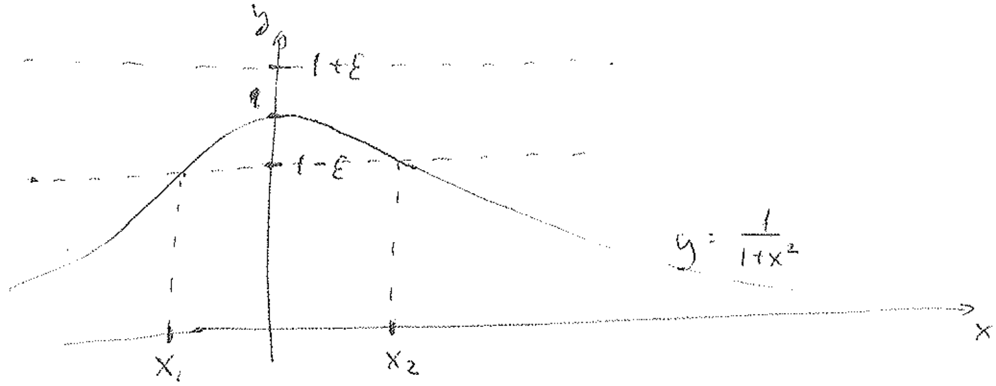
Homework #1, solutions.

Problem 1. (a) Let $\varepsilon > 0$ be given.

1. Coming up with δ.

Consider the graph:



Clearly $x_1 = -x_2$ because the function is symmetric.
If x is between x_1, x_2, then $f(x)$ is between $1-\varepsilon$, $1+\varepsilon$.
So, we should find x_2 and take $\delta = x_2$.
To find x_2, solve

$$\frac{1}{1+x_2^2} = 1-\varepsilon,$$

$$1+x_2^2 = \frac{1}{1-\varepsilon}$$

$$x_2^2 = \frac{1}{1-\varepsilon} - 1 = \frac{\varepsilon}{1-\varepsilon}, \quad x_2 = \sqrt{\frac{\varepsilon}{1-\varepsilon}}.$$

2. Proof that this δ works.

For any $\varepsilon > 0$, let $\delta = \sqrt{\frac{\varepsilon}{1-\varepsilon}}$ (take $\delta = 1$ if $\varepsilon \geq 1$).
We show that:

$$\text{if } 0 < |x - 0| < \delta \quad \text{then} \quad \left|\frac{1}{1+x^2} - 1\right| < \varepsilon.$$

If $|x| < \delta$, then

$$x^2 < \frac{\varepsilon}{1-\varepsilon},$$

$$1+x^2 < 1 + \frac{\varepsilon}{1-\varepsilon} = \frac{1}{1-\varepsilon},$$

$$\frac{1}{1+x^2} > 1-\varepsilon$$

$$\frac{1}{1+x^2} - 1 > -\varepsilon.$$

The inequality

$$\frac{1}{1+x^2} - 1 < \varepsilon$$

follows from

$$\frac{1}{1+x^2} < 1 \ , \quad \frac{1}{1+x^2} - 1 < 0 .$$

Thus, we have shown that

$$-\varepsilon < \frac{1}{1+x^2} - 1 < \varepsilon ,$$

which is equivalent to

$$\left| \frac{1}{1+x^2} - 1 \right| < \varepsilon .$$

The case where $\varepsilon \geq 1$, $\delta = 1$ follows simply because for any x

$$\left| \frac{1}{1+x^2} - 1 \right| < 1 \leq \varepsilon . \qquad \square .$$

Problem 1(b). Let $\varepsilon > 0$ be given.

Since $\lim_{x \to 5} f(x) = 11$, there exist a $\gamma > 0$, such that

if $0 < |x-5| < \gamma$ then $|f(x) - 11| < \varepsilon$.

Because $f(5) = 11$, we may remove the inequality $0 < |x-5|$ and write

$$|x-5| < \gamma \implies |f(x) - 11| < \varepsilon. \qquad (*)$$

Now using the assumption that $\lim_{x \to 3} g(x) = 5$, for the γ above there exists $\delta > 0$, such that

$$0 < |x-3| < \delta \implies |g(x) - 5| < \gamma. \qquad (**)$$

Combining the two implications:

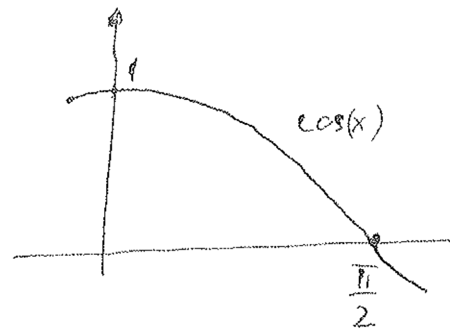
$$0 < |x-3| < \delta \overset{(**)}{\implies} |g(x) - 5| < \gamma \overset{(*)}{\implies} |f(g(x)) - 11| < \varepsilon.$$

Thus, we proved that for any $\varepsilon > 0$ there exist $\delta > 0$, s.t.

if $0 < |x-3| < \delta$ then $|f(g(x)) - 11| < \varepsilon$. $\square$

Problem 2.

(a) Let $a = 0$, $b = \pi/2$.



and define

$$f(x) = \cos(x) - x.$$

Then

$$f(a) = 1 > 0$$
$$f(b) = -\frac{\pi}{2} < 0$$

From the intermediate value theorem, there exists x^* in $(0, \frac{\pi}{2})$, such that $f(x^*) = 0$, which means

$$\cos(x^*) = x^*. \qquad \square$$

(b) Define $g(x) = f(x) - f(x+1)$ on the interval $[0, 1]$.

Now

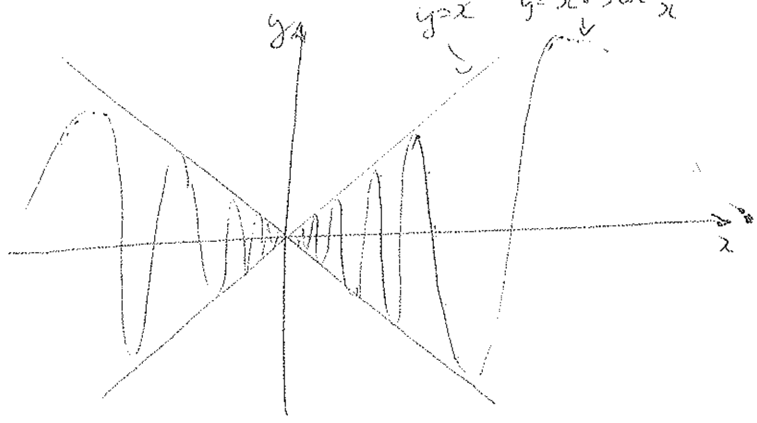
$$g(0) = f(0) - f(1)$$
$$g(1) = f(1) - f(2) = f(1) - f(0) = -g(0).$$

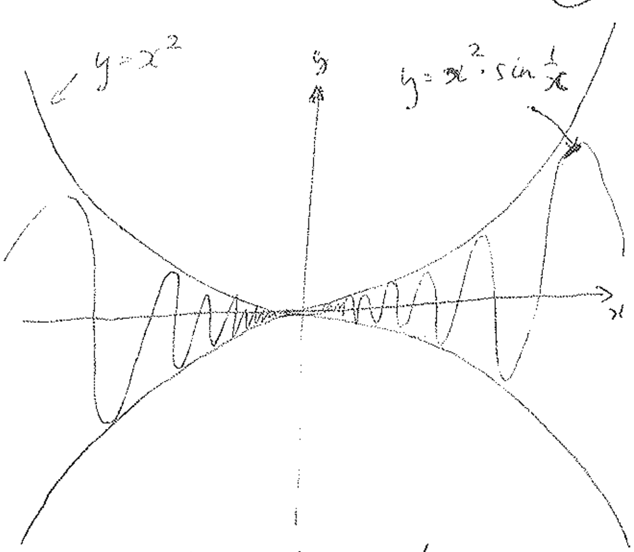
If $g(0)$ is 0, then $x = 0$ satisfies $f(x) = f(x+1)$.

If $g(0) \neq 0$, then $g(0)$ and $g(1)$ have different sign, hence by the IVT, $g(x) = 0$ for some x in $(0,1)$, which means $f(x) = f(x+1)$. $\square$

Important!
$g(x)$ is continuous on the interval $[0,1]$.

Problem 3.





For any $f(x)$, such that $f(0)=0$, the derivative at $a=0$ is by definition

$$f'(0) = \lim \frac{\Delta y}{\Delta x} = \lim_{x\to 0} \frac{f(x)}{x}.$$

In the first case we have

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x \cdot \sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x},$$

which does not exist.

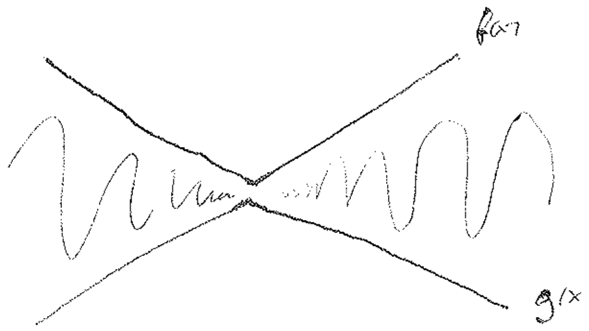
In the second case,

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2 \cdot \sin\frac{1}{x}}{x} = \lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0$$

The limit here exists because the function $x \cdot \sin\frac{1}{x}$ is squeezed between two functions

$$f(x) = |x|, \quad g(x) = -|x|,$$

both of which have the same limit 0:



$\square$.