SCIENCE ONE, MATHEMATICS - HOMEWORK #1

Due 10AM, Friday, Sept. 19

PROBLEM 1. Use the $\varepsilon - \delta$ definition of limits to prove the following.

(a)Show that

$$\lim_{x \to 0} \frac{1}{1+x^2} = 1.$$

(b)Assume that

$$\lim_{x \to 3} g(x) = 5, \quad \lim_{x \to 5} f(x) = 11 = f(5).$$

Show that then

$$\lim_{x \to 3} f(g(x)) = 11.$$

(Hint: argue as follows: we can make $f(g(x))$ ε-close to 11 if the argument $g(x)$ is δ-close to 5. We can also make $g(x)$ δ-close to 5 if x is γ-close to 3.)

PROBLEM 2. Applications of the intermediate value theorem.

(a)Prove that

$$\cos(x) = x$$

for some real number x. You may assume that $\cos x$ is a continuous function.

(b)Let $f(x)$ be a continuous function defined on the interval $[0, 2]$, such that $f(0) = f(2)$. Prove that there exists a real number x in $[0, 1]$, such that $f(x) = f(x+1)$. (Hint: consider the function $g(x) = f(x) - f(x+1)$.)

PROBLEM 3. Using the definition of derivative as the limit of slopes of secant lines, explain why the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

does not have a derivative at $x = 0$, but the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

does have a derivative at $x = 0$. You don't need to give a formal proof here in $\varepsilon - \delta$ notation. It is sufficient to explain in words (and possibly pictures) why the limit in the first case does not exist, but it does exist in the second case.

How to write proofs.
These problems will be graded, therefore write your solutions neatly. Your answers should be readable and not too long. You may also type your solutions.

Please look at the examples 2 and 3 in section 2.4 in textbook for an idea what a proof in the $\varepsilon - \delta$ notation should look like. You can shorten the solutions in these examples by omitting the first part, guessing the value of δ. You proof could start: For any ε, let's choose $\delta = \varepsilon/4$. Then show that this δ works. You do not need to show how you came up with this δ.

Your proof should not be a sequence of formulas only. Write the proof like an essay, with full sentences: If .... then ....., thus ...., therefore ....

A proof should be as short and clear as possible. Writing more stuff than is necessary for the proof only makes it more obscure.