SCIENCE ONE – MATHEMATICS
MIDTERM EXAM
MARCH 3, 2015

Name: SOLUTIONS
Student ID:

Exam rules:

- No calculators, other electronic devices, open books or notes are allowed.
- There are 7 problems in this exam.
- Use opposite empty pages if needed.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		~~80~~ 70

PROBLEM 1. (a)Find the integral

$$\int_0^{\pi/4} \tan^2(x)\sec^4 x dx.$$

$u = \tan(x)$ $\quad x = 0,\ u = 0$

$du = \sec^2(x)\, dx$ $\quad x = \frac{\pi}{4},\ u = 1$

$$\int_0^1 u^2(u^2+1)\, du =$$

$$= \frac{u^5}{5} + \frac{u^3}{3} \Big|_0^1$$

$$= \boxed{\frac{1}{5} + \frac{1}{3}}$$

(b)Solve the differential equation

$$\frac{dy}{dx} = \frac{y \ln y}{x}, \quad y(1) = e^2.$$

$$\int \frac{dy}{y \ln y} = \int \frac{dx}{x} = \ln(x) + C$$

$u = \ln y$

$du = \frac{1}{y} dy$

$$\int \frac{1}{u} du = \ln u = \ln \ln y$$

$$\ln(\ln(y)) = \ln(x) + C$$

$$\ln y = x \cdot e^C$$

$$y = e^{x \cdot B} \qquad y(1) = e^B = e^2 \Rightarrow B = 2$$

$$\boxed{y = e^{2x}}$$

PROBLEM 2. Find the indefinite integral

$$I = \int \frac{\arctan(x)}{x^2}\,dx.$$

$$u = \arctan(x) \qquad dv = \frac{1}{x^2}\,dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = -\frac{1}{x}$$

$$I = -\frac{1}{x}\cdot\arctan(x) + \int \frac{1}{(1+x^2)\cdot x}\,dx$$

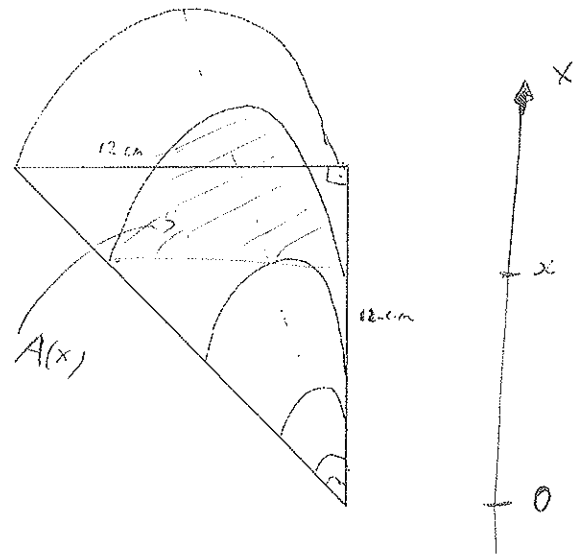
$$\frac{1}{(1+x^2)\cdot x} = \frac{A+Bx}{1+x^2} + \frac{C}{x} = \frac{Ax + Bx^2 + C + Cx^2}{(1+x^2)\cdot x}$$

$$C = 1, \quad A = 0, \quad B = -1$$

$$\int \frac{-x}{1+x^2} + \frac{1}{x}\,dx = -\frac{1}{2}\ln(1+x^2)\ \cancel{\arctan(x)} + \ln(x) + C$$

$$I = \boxed{-\frac{1}{x}\arctan(x) - \frac{1}{2}\ln(1+x^2) + \ln(x) + C}$$

Problem 3. The base of a solid is an isosceles right-angled triangle with equal legs measuring 12 cm. Each cross-section perpendicular to one of these legs is half of a circular disk. Find the volume of the solid.



$$A(x) = \frac{1}{2}\pi R^2$$

$$= \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{\pi}{8}x^2$$

$$\text{Vol} = \int_0^{12} A(x)\,dx$$

$$= \frac{\pi}{8}\int_0^{12} x^2\,dx$$

$$= \frac{\pi}{8}\cdot\frac{x^3}{3}\Big|_0^{12}$$

$$= \frac{\pi}{8}\cdot\frac{12^3}{3}$$

$$= \frac{\pi\cdot 6^3}{3} = 72\pi$$

PROBLEM 4. Find the value of the constant C for which the function f is a probability density function on the given interval.

$$f(x) = Cxe^{-\pi x} \quad \text{on } [0, \infty).$$

Nud:

$$\int_0^{\infty} Cx\, e^{-\pi x}\, dx = 1$$

$$u = Cx \qquad dv = e^{-\pi x}\, dx$$

$$du = C\, dx \qquad v = -\frac{1}{\pi} e^{-\pi x}$$

$$\underbrace{C \cdot x \cdot \left(-\frac{1}{\pi}\right) e^{-\pi \cdot x} \Big|_0^{\infty}}_{= 0} + \frac{C}{\pi} \int_0^{\infty} e^{-\pi x}\, dx$$

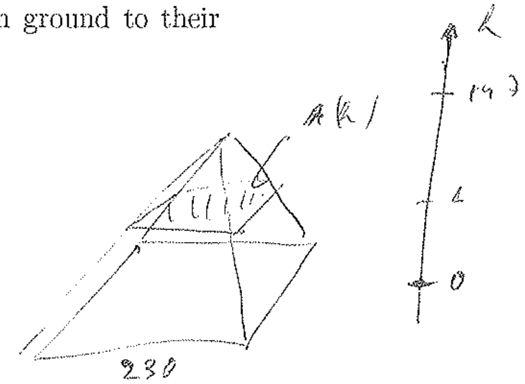
$$\lim_{t \to \infty} -\frac{C}{\pi} t e^{-\pi t} = 0$$

$$= \frac{C}{\pi} \cdot \frac{1}{-\pi} e^{-\pi x} \Big|_0^{\infty}$$

$$= \frac{C}{\pi^2}$$

$$\frac{C}{\pi^2} = 1 \quad \Rightarrow \quad \boxed{C = \pi^2}$$

PROBLEM 5. The Great Pyramid of Giza was built of limestone in Egypt over a 20-year time period (from 2580 BC to 2560 BC). Its base is a square with side length 230 metres and its height when built was 147 metres. The density of the limestone is about $2400 kg/m^3$. Assuming the pyramid has smooth sides, find an integral that computes the total work done in building the pyramid. (That means, the work of lifting the limestone blocks from ground to their position in the pyramid.) Do not evaluate the integral.



$$A(h) = \left(230 - \frac{230}{147} h\right)^2$$

$$W = \int_0^{147} A(h) \cdot h \cdot g \cdot 2400 \, dh$$

PROBLEM 6. Find x such that the integral

$$F(x) = \int_x^{x+1} \frac{1}{t^2+t+1}\,dt$$

is ~~minimal~~.

maximal

$$F'(x) = \frac{1}{(x+1)^2+(x+1)+1} - \frac{1}{x^2+x+1} = 0$$

$$(x+1)^2+(x+1)+1 = x^2+x+1$$

$$x^2+2x+1+x+1+1 = x^2+x+1$$

$$2x+2 = 0$$

$$\boxed{x=-1}$$

PROBLEM 7. Find the integral

$$I = \int_1^\infty \frac{1}{x} - \frac{1}{\sqrt{x^2-1}} dx.$$

(Hint: Do not separate it into two integrals. Integrate it as one integral from beginning to end.)

$$\sec^2(u) - 1 = \frac{1}{\cos^2(u)} - 1 = \frac{1-\cos^2(u)}{\cos^2 u} = \tan^2(u)$$

$$x = \sec(u)$$

$$dx = \sec(u)\tan(u)\, du$$

$$I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sec(u)} - \frac{1}{\tan(u)} \right) \sec(u) \cdot \tan(u)\, du$$

$\sec(u) = x = 1 \quad \cancel{\sec(u)} = 0$

$\sec u = \infty \quad u = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \tan(u) - \sec(u)\, du$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(u) - 1}{\cos(u)}\, du$$

$$v = \sin(u)$$

$$dv = \cos(u)\, du$$

$$= \int_0^1 \frac{v-1}{1-v^2}\, dv$$

$$= -\int_0^1 \frac{1}{1+v}\, dv$$

$$= -\ln(1+v) \Big|_0^1$$

$$= \boxed{-\ln(2)}$$