Problem 1. Let \( C \subset IR^{m+n} \) be the cone generated by 
\[ \{ e_i + e_j \} \quad i = 1, \ldots, p \]
\[ j = p+1, \ldots, p+n \].

Then the ring is the semigroup ring \( C[IR^{m+n}] \).

Problem 2. Let \( \sigma \) be the given cone in \( N \subset Z^n \).

(i). Let \( N' \subset N \) be the lattice generated by
\[ e_1, \ldots, e_n, \ e_i - e_j \quad i \neq j \].

Then \((\sigma, N')\) is nonsingular. The dual cone is
\((\sigma^*, M')\), where \( M' \) is generated by \( v_i^*, \ldots, v_n^* \).

Notice: \( M \subset M' \)
\[ v_i^* = e_i^* + \frac{1}{m} e_n^* \]
\[ v_n^* = \frac{1}{m} e_n^* \]

Let \( z_i = x_i v_i^* \). Then \( N/N' \subset Z_m.Z \) generated by \( e_n \)
acts on \( k[2, \ldots, 2n] \)
\[ e_n \cdot x_i v_i^* = e^{2 \pi i \cdot \langle e_n, v_i^* \rangle} \cdot x_i v_i^* = f_m \cdot x_i v_i^* \]

The quotient by this action is \( M_\sigma \).

(ii) The Veronese embedding of \( IR^{m+n} \) is
\[ \text{Proj}^n \oplus C[IR^{2m+1}] \] mod degree and part.
This ring is precisely the \( m \)-invariants of \( k[2, \ldots, 2n] \).
Problem 3.

Let $N \subseteq \mathbb{Z}^2$ be a lattice such that the fan is nonsingular in $N$. Let $v_i \in N$ be the first lattice points on rays.

In a nonsingular fan:

$$v_1 + v_3 = a_2 v_2, \quad a_2 \in \mathbb{Z}.$$ 

For the fan $\Delta$, we get

$$v_1 + v_3 = a_2 v_2 \Rightarrow v_1 = (-m, c), \quad v_3 = (c_1 - m), \quad v_2 = (p, p) \quad p | m.$$ 

$$v_1 + v_3 = a_4 v_4 \Rightarrow (-m_1 - m) = a_4 \cdot b \cdot (2, 3), \quad \text{contradiction.}$$
Problem 4

1. Write

\[ v_j = -av_i + bv_{i+1} \]
\[ v_{j+1} = -cv_i - dv_{i+1} \]

We need

\[ \det \begin{bmatrix} v_j & v_{j+1} \end{bmatrix} = \begin{vmatrix} -a & -c \\ -b & -d \end{vmatrix} = ad + bc = 1 \]

2. Choose a basis for \( N \) so that \( v_1 = (1,0) \), \( v_2 = (1,1) \).

Prove: When \( d = 3 \), then

\[ \begin{cases} v_1 + v_3 = a \cdot v_2 \\ v_2 + v_3 = b \cdot v_1 \end{cases} \]

This implies \( a = b = -1 \), \( v_3 = (-1,-1) \).

When \( d = 4 \), we get

\[ v_3 = v_2 + v_1 \]

This implies \( a = -b \), \( c = -d \). Both \( a \) and \( c \) cannot be zero because then \( v_2 = -v_1 \), \( v_3 = -v_1 \), \( v_1 + v_3 = 0 \).

If \( c = 0 \), \( a \in \mathbb{Z} \), we get \( F_4 \).

3. (a) If \( v_i \neq v_1 \) for any \( i \), then there are 3 vectors in the same half-plane:

Assume no more rays have

\[ \begin{bmatrix} v_{i+2} \\ v_{i+1} \end{bmatrix} \]

no rays in halfplane except \( v_{i+3} \).
(b) Let \( v_0 = v_i \)

\[
\begin{align*}
v_j &= -b_j v_0 + b_j v_i, & j = 1, \ldots, i, \quad b_j, b_j' > 0, \\
c_j &= b_j + b_j' \\
c_i &= 1, \quad c_i = 1 \\
c_2 &> 2.
\end{align*}
\]

Choose \( j \) s.t. \( c_j > c_{j-1}, \quad c_j > c_{j-1} \). Then

\[
\begin{align*}
a \cdot v_j &= v_{j-1} + v_{j+1}, & a \in \mathbb{R} \\
a \cdot c_j &= c_{j-1} + c_{j+1} \quad \Rightarrow \quad 1 \leq a < 2, \quad a = 1.
\end{align*}
\]

4. The product maps \( v_0 \to v_0, \quad v_i \to v_i \).

5. The first claim is clear. The equality holds for the fan of \( \mathbb{R}^2 \) and \( F \), if it is preserved by a star subdivision.

6. For a toric surface with \( d \) rays,

\[
\begin{align*}
X(\mathcal{O}_X) &= 1 \\
K_X \cdot K_X &= \sum_{i,j=1}^{d} D_i \cdot D_j = 2d - \sum_{i \in \mathcal{R}} a_i \\
X(X) &= d
\end{align*}
\]

RR:

\[
i = \frac{1}{12} \left( 2d - \sum a_i + d \right).
\]