MATH 615B - HOMEWORK #2

Due Friday, Oct. 13

Problem 1. Exe 3 on page 31. (Show that the ring is $A_\sigma$ for some cone $\sigma$ in a lattice $N$. All such ring are Cohen-Macaulay. It is not hard to write down the cone $\sigma^\vee$, but you need to show that the given monomials generate the semigroup.)

Problem 2. Exe 1 on page 35. Recall that the coordinate ring of $\mathbb{C}^n / G$ is the ring of $G$-invariant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.

Problem 3. Exe 1 on page 36.

Problem 4. All problems except the last one in Section 2.5. These problems involve simple combinatorics. Many of them have hints at the end of the book. The goal is to classify all nonsingular complete toric surfaces.

Problem 5. Read the last problem in Section 2.5 and use its statement to interpret the formula (**) as the Riemann-Roch formula.

For a nonsingular complete surface $X$, the Riemann-Roch formula applied to the sheaf $\mathcal{O}_X$ reads:

$$\chi(\mathcal{O}_X) = \frac{1}{12} (K_X \cdot K_X + \chi(X)),$$

where

• $\chi(\mathcal{O}_X)$ is the algebraic Euler characteristic:

$$\chi(\mathcal{O}_X) = \sum_i (-1)^i \dim H^i(X, \mathcal{O}_X).$$

We will see later that for complete toric varieties (similarly to the case of $\mathbb{P}^n$), the sheaf $\mathcal{O}_X$ has no higher cohomology, hence

$$\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) = 1.$$

• $K_X$ is the canonical divisor of $X$. For a toric surface it is equal to $K_X = \sum_i -D_i$, where $D_i$ are the curves from the problem.

• $\chi(X)$ is the topological Euler characteristic of $X$ as a complex manifold:

$$\chi(X) = \sum_i (-1)^i \dim H^i(X, \mathbb{C}).$$

When a torus $T$ acts on $X$ with finitely many fixed points then $\chi_X$ is simply the number of $T$-fixed points (because $T$ and any nontrivial $T$-orbit has Euler characteristic 0.) The $T$-fixed points are where the curves $D_i$ meet pairwise.