Problem 1. Generators of $\sigma$ map to generators of $\sigma'$.
Support hyperplanes of $\sigma'$ pull back to support hyperplanes of $\sigma$. This gives a map (injective).
\[ \{ \text{Faces of } \sigma \} \to \{ \text{Faces of } \sigma' \} \]
Every such pullback contains $W$, hence the face of $\sigma$ contains $W$.

Problem 2. Let $\sigma'$ be defined by $\{ u \in \mathbb{R}^k \mid u \cdot \mathbf{v} = 0 \}$, $\mathbf{v} \in \mathbb{R}^k$. If $v_1, v_2 \in \mathbb{R}$, $v_1, v_2 \in \mathbb{R}$, then $0 = u(v_1 + v_2) = u(v_1) + u(v_2)$, hence both must be zero, $v_1, v_2 \in \mathbb{R}$.

Conversely, let $\sigma'$ be the smallest face of $\sigma$ containing $\mathbf{v}$. Replace $\sigma$ by $\sigma'$ and assume $\mathbf{v}$ does not lie in a proper face of $\sigma$. Then $\mathbf{v}$ intersects the interior of $\sigma$. If $\mathbf{v} \notin \mathbf{v}$, let $v_1, ..., v_n$ be minimal generators for $\sigma$. Any point $p$ in the can be written as
\[ p = \sum a_i v_i \]
where all $a_i > 0$. If $p \in \mathbf{v}$, this implies all $v_i \in \mathbf{v}$, $\mathbf{v} = \mathbf{v}$.

Problem 3. Let $m_1, ..., m_n$ generate $\sigma$. Then they also generate $M$ as an abelian group. Let $\sigma' = \langle m_1, ..., m_n \rangle \leq \mathbb{R} M_{\mathbb{R}}$. We claim that $\sigma = \sigma' \cap M$. Clearly $\sigma \subseteq \sigma' \cap M$. Conversely, if $m \in \sigma' \cap M$, $m = \sum \lambda_i m_i$, $\lambda_i \in \mathbb{R}_{>0}$.

Thus for some $c > 0$, $c m_i \in \mathbb{Z}$, hence $c \cdot m \in \sigma$. This implies $m \in \sigma$.

Problem 4.
(a) We have the surjective map
\[ C[x,y]/(x^2 - y^3) \twoheadrightarrow C[x^2, y^3] \]
Let's find its kernel. Modulo $x^2 - y^3$, an element in $C[x,y]$ can be written as $p(y) + yq(y)$ for $p, q$ polynomials. This element maps to $p(x^2) + x^2 q(x^2)$, which is 0 iff $pq = 0, qp = 0$.

(b) The same argument gives $C[x^2, y^3] \cong C[x,y]/(x^2 - y^3)$. 


Problem 5.

If \(..., u_n \in S^\nu M$ is a finite set, then $\langle u_1, ..., u_n \rangle$ is a rational subcon of $S^\nu$. There are rational points in $S^\nu$ that do not lie in this subcone.

\[ \langle u_1, ..., u_n \rangle \leq C[S^\nu M] \]

Let $I$ be the ideal generated by all monomials $x^{u_i}$. This ideal is not finitely generated. If it is generated by $f_1, ..., f_n$, we may replace each $f_i$ by monomials in it, hence assume the ideal is generated by monomials $x^{u_1}, ..., x^{u_n}$.

Picture: ideal in $k[x,y]$ generated by monomials.

Similarly in $C[S^\nu M]$:

monomial not in $\langle x^{u_1}, ..., x^{u_n} \rangle$

monomials in $\langle x^{u_1}, x^{u_2}, x^{u_3} \rangle$
Problem 6

\[ X_\Delta = P^1 \times P^1 \setminus \{A, B\} \]

How to get this: Consider \( P^1 \times P^1 \)

Each \( M_{\sigma_i} = A^2 \), these cover \( P^1 \times P^1 \)

\[ M_{\sigma_i} = P^1 \times P^1 \setminus 2 \text{ lines} \]

The \( X_\Delta \) above are unions of some \( M_{\sigma_i} \) and some \( M_\sigma \).

\[ X_\Delta = A^2 \setminus \{(0,0)\} \]
Problem 7.

(1) \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \):

Graph of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \),

- \( f \) is linear on each cone \( \sigma \) of \( \mathbb{P}^1 \).
- \( f \) maps lattice points to lattice points.

\( \mathbb{P}^1 \) - bundle over \( \mathbb{P}^1 \):

\[ \Delta: \]

Graph of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), linear on each cone, integral.

Fan of \( \mathbb{P}^1 \):

Let \( \sigma_1, \sigma_2 \) be the rays of the fan of \( \mathbb{P}^1 \).
Maximal cones of the fan \( \Delta \) are \( \sigma_i \) and \((\text{graph of } f |_{\sigma_i}) \circ \sigma_i \) for each cone in the fan of \( \mathbb{P}^1 \).

(2) Use a function \( f : \mathbb{R}^1 \rightarrow \mathbb{R}^2 \) s.t. \( f \) is linear on each cone and maps lattice points to lattice points.
Problem 8.

(1) \( \Delta = \mathbb{A}^3 \setminus \{0\} \).

(2) Project

\[ \Pi: \mathbb{N} \longrightarrow \mathbb{N}^1 = \mathbb{N}/(1,1) \mathbb{Z} \]

For every \( \sigma \in \Delta \), we can split

\[ (\sigma, N) = (\sigma, \text{span} \sigma \cap N) \times (0, (1,1) \mathbb{Z}) \]

with \( \Pi \) the projection to the first factor.

The fibers are \( \mathbb{C}^x \), the image is \( \mathbb{P}^2 \).

This map is the quotient

\[ \mathbb{A}^3 \setminus \{0\} \longrightarrow \mathbb{A}^3 \setminus \{0\} / \mathbb{C}^x = \mathbb{P}^2. \]