All numbers refer to problems in Fulton’s book. The exercises in the book are not numbered, so I call them “Exercise 2 on page 54”, for example. Some exercises have hints at the end of the book. All exercises in this homework come from Chapter 1.

PROBLEM 1. Ex 1 on page 14. (Here $R\tau$ is the span of $\tau$. This problem should remind you of one of the isomorphism theorems for groups.)

PROBLEM 2. Ex 1 on page 15.

PROBLEM 3. Ex 1 on page 19.

PROBLEM 4. Ex 1 on page 20 (a,b). (Such nonsaturated semigroups corresponding to non-normal varieties will not be used later.)

PROBLEM 5. Ex 2 on page 20. (Hint: A finite set of lattice points generates a rational subcone. For non-noetherian property, show that the ideal generated by all monomials except 1 is not finitely generated. If this ideal were generated by a finite set of polynomials, we may assume that these polynomials are in fact monomials. What is the set of monomials in this ideal?)

PROBLEM 6. Ex 3 on page 22. (You may assume the result of Ex 3 on page 19. We will also talk about this in class.)

The next two problems study toric varieties that are locally products of two toric varieties. A simple example of this is the Hirzebruch surface $F_a$. Consider the Hirzebruch surface $F_a$ with fan $\Delta_a$ as shown in the second picture on page 7. The projection of the fan to the $x$-axis defines a toric morphism $F_a \to \mathbb{P}^1$. We can cover $\mathbb{P}^1$ with two open sets $U_1 \simeq \mathbb{A}^1$ and $U_2 \simeq \mathbb{A}^1$, corresponding to the two rays of the fan. Similarly, we can write the fan $\Delta_a$ as a union of two fans, the inverse images of the two rays. This expresses $F_a = V_1 \cup V_2$, where the open set $V_i$ maps to $U_i$, $i = 1, 2$. Now $V_i$ is isomorphic to $U_i \times \mathbb{P}^1$, with the map being the projection to $U_i$. This can be seen by writing the fan of $V_i$ as a product of two fans. Hence the map $F_a \to \mathbb{P}^1$ has fibers isomorphic to $\mathbb{P}^1$.

PROBLEM 7. The Hirzebruch surface $F_a$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$.

1) Describe a toric construction of $\mathbb{P}^1$ bundles over $F_a$ (or a general toric variety) as follows. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function that is linear when restricted to every cone of $\Delta_a$ and takes integer values at integer points. Explain why each such function $f$ corresponds to a $\mathbb{P}^1$ bundle over $F_a$.

2) How would one construct a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$? A $\mathbb{P}^2$ bundle over $F_a$?
Problem 8. Let $N = \mathbb{Z}^3$, let $\sigma$ the positive octant in $\mathbb{R}^3$, and let $\Delta$ be the fan consisting of all proper faces of $\sigma$

$$\Delta = \{ \tau \leq \sigma | \tau \neq \sigma \}.$$

(1) Describe the variety $X(\Delta)$. (It should be an open subvariety of $U_{\sigma}$ because all faces $\tau \leq \sigma$ give open subsets of $U_{\tau} \subseteq U_{\sigma}$.)

(2) Let $\pi : N \to N' \simeq \mathbb{Z}^2$ be the projection from $(1, 1, 1)\mathbb{Z}$, mapping $\Delta$ to a fan $\Delta'$ in $N'$. Describe the fibres of the corresponding projection of toric varieties. What is the image variety?