Some problems below ask about irreducibility and dimension of a variety. You can use the following results, some of which we have not proved:

1. We will consider the dimension of an irreducible variety. Both $\mathbb{A}^n$ and $\mathbb{P}^n$ have dimension $n$.

2. Let $f : Y \to X$ be a morphism of projective varieties. If $Y$ is irreducible then both $X$ and the general fibre of $f$ are irreducible. The dimension of $Y$ is the dimension of $X$ plus the dimension of the general fibre. (Some fibres may be reducible and have higher dimension than the general fibre. This happens for example in case of a blowup map.)

3. Conversely, if $f : Y \to X$ is a surjective morphism as above and if $X$ and all fibres of $f$ are irreducible, the $Y$ is also irreducible.

Recall that the Grassmannian $Gr(m, n)$ is the set of $m$-dimensional subspaces of $k^n$. We can also view it as $G(m, n)$, the set of $m$-1 planes in $\mathbb{P}^{n-1}$. The Plücker morphism

$$Gr(m, n) \to \mathbb{P}(\Lambda^m k^n)$$

embeds $Gr(m, n)$ in $\mathbb{P}^{\binom{n}{m}-1}$ as a closed subvariety. The Grassmannian $Gr(2, 4) = G(1, 3)$ is embedded in $\mathbb{P}^5$ as a degree 2 hypersurface defined by the Plücker relation

$$Z_{1,2}Z_{3,4} - Z_{1,3}Z_{2,4} + Z_{1,4}Z_{2,3} = 0.$$

**Problem 1.** Let $G = G(1, 3)$ be the Grassmannian of lines in $\mathbb{P}^3$. For a point $p \in \mathbb{P}^3$, define $X_p \subset G$ as the set of lines through $p$.

1. Prove that $X_p$ maps to a 2-plane in $G$ by the Plücker embedding. (Hint: it does not matter which point $p$ you take. You can fix one specific $p$, for example $p = (1 : 0 : 0 : 0)$ and find $X_p$.)

2. Prove that $X_p \cap X_q$ is a point if $p \neq q$. In particular, $X_p \neq X_q$.

3. Prove that the planes $X_p$ cover $G$.

(a) We can take $w_1 = p = (1, 0, 0, 0)$. If $w_2 = (a, b, c, d)$, then $w_1 \wedge w_2$ has nonzero coordinates $b, c, d$.

(b) There is a unique line through $p$ and $q$.

(c) Every line passes through some point.

**Problem 2.** Let $G = G(1, 3)$ be as before. For a 2-plane $W \in \mathbb{P}^3$, define $X_W \subset G$ as the set of lines through $W$.

1. Prove that $X_W$ maps to a 2-plane in $G$ by the Plücker embedding. (You may again fix one specific plane.)

2. Prove that $X_W \cap X_V$ is a point if $V \neq W$.

3. Prove that the planes $X_W$ cover $G$.

4. Prove that $X_W \cap X_p$ is either empty or a line. In particular, $X_W \neq X_p$.

(a) The image of $X_W$ is the projectivization of $\Lambda^2 W \subseteq \Lambda^2 k^4$.

(b) Two planes intersect along a unique line.
Every line lies on some plane.

\(X_W \cap X_p\) is empty if \(p \notin W\). Otherwise it is the set of lines on \(W\) containing \(p\).

This is a \(\mathbb{P}^1\) in \(G\).

**Problem 3.** Let \(X \subset \mathbb{P}^n\) be an irreducible projective variety.

(a) Define the incidence variety
\[
I_m(X) = \{W \in \mathbb{G}(m, n)|W \cap X \neq \emptyset\}.
\]
Prove that \(I_m(X)\) is a closed and irreducible subset of \(\mathbb{G}(m, n)\). (Hint: Use the universal \(m\)-plane.)

(b) Define the Fano variety of \(X\)
\[
F_m(X) = \{W \in \mathbb{G}(m, n)|W \subset X\}.
\]
Prove that \(F_m(X)\) is a closed subset of \(\mathbb{G}(m, n)\). Find an example where it is reducible.

(a) The universal \(m\)-plane projects to \(\mathbb{G}(m, n)\) and to \(\mathbb{P}^n\). Let \(Y\) be the inverse image of \(X\) by the second projection. It has fibres \(m\)-planes through a point. These fibres are all isomorphic to \(\mathbb{G}(m-1, n-1)\). This shows that \(Y\) is a closed irreducible subset of the universal plane. The first projection is proper, it maps \(Y\) to the closed irreducible set \(I_m(X)\).

(b) Consider the variety \(Y\) in the previous part as a closed subvariety of the universal \(m\)-plane. The Fano variety is the set of points in \(\mathbb{G}(m, n)\) such that the fibre over it is the whole \(m\)-plane. We may cover the Grassmannian with open affines \(U\) such that the universal \(m\)-plane over \(U\) is a product \(\mathbb{P}^m \times U\). The subvariety \(Y\) is defined by homogeneous polynomials with coefficients in \(A(U)\). Vanishing of the coefficients defines the Fano variety.

The quadric \(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3\) is covered by two families of lines. Hence its Fano variety of lines consists of two copies of \(\mathbb{P}^1\). The Grassmannian \(\mathbb{G}(1, 3)\) is also covered by two families of planes. More generally, a smooth quadric in \(\mathbb{P}^{2m+1}\) contains two families of \(m\)-planes.

**Problem 4.** Let \(X, Y \subset \mathbb{P}^n\) be irreducible projective varieties, \(X \cap Y = \emptyset\). Define their join \(J(X, Y)\) as the union of all lines intersecting both \(X\) and \(Y\).

(a) Prove that \(J(X, Y)\) is closed in \(\mathbb{P}^n\).

(b) Prove that \(J(X, Y)\) is irreducible.

(c) Fix \(X\) and \(Y\) to be your favourite two skew lines in \(\mathbb{P}^4\) and find an equation defining their join.

(a) Consider the morphism
\[
X \times Y \times \mathbb{P}^1 \to \mathbb{P}^n,
\]
sending \((x, y, (s : t))\) to \(sx + ty\). The image of this map is the join. Since the domain is projective and irreducible, the morphism has closed irreducible image.

(b) Proved in the previous part.

(c) Two skew lines in \(\mathbb{P}^4\) lie in a hyperplane \(H \cong \mathbb{P}^3\). The join must also lie in \(H\), and is in fact equal to the hyperplane.

**Problem 5.** Let \(X \subset \mathbb{P}^n\) be a hypersurface of degree \(d\). Consider the Fano variety of lines in \(X\):
\[
F(X) = \{[L] \in \mathbb{G}(1, n)|L \subset X\}.
\]
The problem here is to find the dimension of $F(X)$ for a general hypersurface $X$. Let $\Pi = \mathbb{P}^N$ be the parameter space of all degree $d$ hypersurfaces in $\mathbb{P}^n$. (Recall that $\Pi$ is the projectivization of the space of degree $d$ homogeneous polynomials in $X_0, \ldots, X_n$. Here $N = (\binom{n+d}{n} - 1)$.) Let

$$I = \{([L], [X]) \in \mathbb{G}(1, n) \times \Pi | L \subset X\}.$$

You may assume that $I \subset \mathbb{G}(1, n) \times \Pi$ is a closed subset.

(1) Prove that $I$ is irreducible and find its dimension. (Hint: consider the fibres of the projection $I \to \mathbb{G}(1, n)$.)

(2) Assuming that $F(X) \neq \emptyset$ for any $[X] \in \Pi$, find the dimension of $F(X)$ for general $X$.

(3) For each $n$, find the smallest $D$ such that a general hypersurface $X$ of degree $d > D$ contains no line. What can you say about the number of lines on a general hypersurface of degree $D$? (Is it infinite/finite? Can it be zero?)

(1) Given a line $L \subset \mathbb{P}^n$, say $L = \{(s : t : 0 : \ldots : 0)\}$, a hypersurface $V(f(X_0, \ldots, X_n))$ contains $L$ if the coefficients of all monomials in $X_0$ and $X_1$ vanish. This gives $d + 1$ linear conditions. It implies that $I$ is irreducible and has codimension $d + 1$.

(2) If the projection to $\Pi$ is surjective then the general fibre has dimension

$$\dim I - \dim \Pi = \dim \mathbb{G}(1, n) - (d + 1) = 2(n - 1) - (d + 1) = 2n - d - 3.$$

(3) If the general fibre has negative dimension, then it is empty: $D = 2n - 3$. If the degree is equal to $D$, then the general hypersurface contains a finite number of lines.