MATH 532 - HOMEWORK #5

Due Tuesday, Nov. 26

Some problems below ask about irreducibility and dimension of a variety. You can use the following results, some of which we have not proved:

1. We will consider the dimension of an irreducible variety. Both $\mathbb{A}^n$ and $\mathbb{P}^n$ have dimension $n$.
2. Let $f : Y \rightarrow X$ be a morphism of projective varieties. If $Y$ is irreducible then both $X$ and the general fibre of $f$ are irreducible. The dimension of $Y$ is the dimension of $X$ plus the dimension of the general fibre. (Some fibres may be reducible and have higher dimension than the general fibre. This happens for example in case of a blowup map.)
3. Conversely, if $f : Y \rightarrow X$ is a surjective morphism as above and if $X$ and all fibres of $f$ are irreducible, the $Y$ is also irreducible.

Recall that the Grassmannian $Gr(m, n)$ is the set of $m$-dimensional subspaces of $k^n$. We can also view it as $G(m − 1, n − 1)$, the set of $m − 1$ planes in $\mathbb{P}^{n−1}$. The Plücker morphism

$$Gr(m, n) \rightarrow \mathbb{P}(\Lambda^m k^n)$$

$$\text{Span}\{w_1, \ldots, w_m\} \mapsto w_1 \wedge \ldots \wedge w_m.$$ 

embeds $Gr(m, n)$ in $\mathbb{P}^{\binom{n}{m}}$ as a closed subvariety. The Grassmannian $Gr(2, 4) = G(1, 3)$ is embedded in $\mathbb{P}^{5}$ as a degree 2 hypersurface defined by the Plücker relation $Z_{1,2}Z_{3,4} − Z_{1,3}Z_{2,4} + Z_{1,4}Z_{2,3} = 0$.

**Problem 1.** Let $G = G(1, 3)$ be the Grassmannian of lines in $\mathbb{P}^3$. For a point $p \in \mathbb{P}^3$, define $X_p \subset G$ as the case of lines through $p$.

(a) Prove that $X_p$ maps to a 2-plane in $G$ by the Plücker embedding. (Hint: it does not matter which point $p$ you take. You can fix one specific $p$, for example $p = (1 : 0 : 0 : 0)$ and find $X_p$.)

(b) Prove that $X_p \cap X_q$ is a point if $p \neq q$. In particular, $X_p \neq X_q$.

(c) Prove that the planes $X_p$ cover $G$.

**Problem 2.** Let $G = G(1, 3)$ be as before. For a 2-plane $W \in \mathbb{P}^3$, define $X_W \subset G$ as the set of lines through $W$.

(a) Prove that $X_W$ maps to a 2-plane in $G$ by the Plücker embedding. (You may again fix one specific plane.)

(b) Prove that $X_W \cap X_V$ is a point if $V \neq W$.

(c) Prove that the planes $X_W$ cover $G$.

(d) Prove that $X_W \cap X_p$ is either empty or a line. In particular, $X_W \neq X_p$.

**Problem 3.** Let $X \subset \mathbb{P}^n$ be an irreducible projective variety.

(a) Define the incidence variety

$$I_m(X) = \{ W \in \mathbb{G}(m, n) | W \cap X \neq \emptyset \}.$$ 

Prove that $I_m(X)$ is a closed and irreducible subset of $\mathbb{G}(m, n)$. (Hint: Use the universal $m$-plane.)

(b) Define the the Fano variety of $X$

$$F_m(X) = \{ W \in \mathbb{G}(m, n) | W \subset X \}.$$
Prove that $F_m(X)$ is a closed subset of $G(m,n)$. Find an example where it is reducible.

**Problem 4.** Let $X, Y \subset \mathbb{P}^n$ be irreducible projective varieties, $X \cap Y = \emptyset$. Define their join $J(X,Y)$ as the union of all lines intersecting both $X$ and $Y$.

(a) Prove that $J(X,Y)$ is closed in $\mathbb{P}^n$.
(b) Prove that $J(X,Y)$ is irreducible.
(c) Fix $X$ and $Y$ to be your favourite two skew lines in $\mathbb{P}^4$ and find an equation defining their join.

**Problem 5.** Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$. Consider the Fano variety of lines in $X$:

$$F(X) = \{[L] \in G(1,n) | L \subset X\}.$$

The problem here is to find the dimension of $F(X)$ for a general hypersurface $X$. Let $\Pi = \mathbb{P}^N$ be the parameter space of all degree $d$ hypersurfaces in $\mathbb{P}^n$. (Recall that $\Pi$ is the projectivization of the space of degree $d$ homogeneous polynomials in $X_0, \ldots, X_n$. Here $N = \binom{n+d}{n} - 1$.) Let

$$I = \{([L],[X]) \in G(1,n) \times \Pi | L \subset X\}.$$

You may assume that $I \subset G(1,n) \times \Pi$ is a closed subset.

(1) Prove that $I$ is irreducible and find its dimension. (Hint: consider the fibres of the projection $I \to G(1,n)$.)

(2) Assuming that $F(X) \neq \emptyset$ for any $[X] \in \Pi$, find the dimension of $F(X)$ for general $X$.

(3) For each $n$, find the smallest $D$ such that a general hypersurface $X$ of degree $d > D$ contains no line. What can you say about the number of lines on a general hypersurface of degree $D$? (Is it infinite/finite? Can it be zero?)