MATH 532 - HOMEWORK #4

Solutions

Problem 1. Let \( X = Z(y^2 - g(x)) \subset \mathbb{A}^2 \) where \( g(x) \) is a polynomial of degree 3, and let \( \pi : (x,y) \mapsto x \) be the projection. The projective closure \( \overline{X} \subset \mathbb{P}^2 \) is defined by the homogenization of the polynomial \( y^2 - g(x) \). (Set \( x = X_1/X_0, \ y = X_2/X_0 \) and clear denominators in the rational function.)

1. Find all points at infinity \( \overline{X} \setminus X \) and show that \( \pi \) extends to a regular map \( \overline{\pi} : \overline{X} \to \mathbb{P}^1 \). (Hint: the map is \( \overline{\pi} : (X_0 : X_1 : X_2) \mapsto (X_0 : X_1) \). You can change the right hand side to \( (X_0X_1^2 : X_1^3) \) and simplify.)

2. Repeat the previous part with \( X = Z(y^3 - g(x)) \), with \( g(x) \) as before.

(a) When we homogenize the polynomial \( y^2 - g(x) \), assuming \( g \) is monic, we get \( X_2^2X_0 - X_1^3 + X_0(...) \). The only point at infinity \( X_0 = 0 \) is \( (0 : 0 : 1) \). We can change the map to \( (X_0 : X_1) = (X_0X_1^2 : X_1^3) = (X_0X_1^2 : X_2^2X_0 - X_0(...)) = (X_1^2 : X_2^2 - h(X_0, X_1)) \).

This map takes \( (0 : 0 : 1) \) to \( (0 : 1) \).

(b) The homogeneous equation now is \( X_2^3 - X_1^3 + X_0h(X_0, X_1) \). There are now three points at infinity, the three solutions to \( X_2^3 - X_1^3 \). They all have \( X_1 \neq 0 \), hence they are all mapped to \( (0 : 1) \).

Problem 2. Consider the twisted cubic \( C = \{(t,t^2,t^3) | t \in k \} \subset \mathbb{A}^3 \).

Its closure \( \overline{C} \subset \mathbb{P}^3 \) is the image of the Veronese map \( v_3 : \mathbb{P}^1 \to \mathbb{P}^3 \).

(a) The curve \( C \) can be defined by equations \( y = x^2, z = x^3 \). Prove that the homogenizations of these equations do not define the closure \( \overline{C} \). Find another equation \( E \) for \( C \), so that the homogenizations of \( E \) and the other two equations define \( \overline{C} \).

(b) Show that \( C = V(y-x^2, z^2-2xyz+y^3) \) and \( \overline{C} \) is defined by the homogenizations of the two polynomials, but the ideal in \( k[X_0, X_1, X_2, X_3] \) generated by the two homogenized polynomials is not radical.

(c) Consider the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \), defined by \( Z_{01}Z_{10} = Z_{00}Z_{11} \). Prove that \( \overline{C} \) lies on \( \mathbb{P}^1 \times \mathbb{P}^1 \) (with the correct choice of coordinates) and it can be defined by one bi-homogeneous polynomial. Determine the bi-degree of this polynomial.

(a) The homogenizations of the two equations are \( X_0X_2 = X_1^2 \) and \( X_0^2X_3 = X_1^3 \). The line \( X_0 = X_1 = 0 \) satisfies these equations. The homogenization of \( y^3 = z^2 \) is \( X_2^3 = X_0X_3^2 \). The only point on the line \( X_0 = X_1 = 0 \) where this equation is satisfied is \( (0 : 0 : 1) \). This point lies on \( \overline{C} \).

(b) When we substitute \( y = x^2 \) into the second polynomial, we get \((z-x^3)^2\). Thus, the two polynomials vanish precisely on the curve \( C \). The homogenizations of the polynomials are \( X_0X_0 - X_1^2, X_3^2X_0 - 2X_1X_2X_3 + X_2^3 \). We need to check that these don’t vanish at any extra points at infinity. When \( X_0 = 0 \), then also \( X_1 = X_2 = 0 \). The point \( (0 : 0 : 0 : 1) \) lies on \( \overline{C} \).
The polynomial $z - x^3$ vanishes on $C$, hence its homogenization $X_3X_0^3 - X_1^3$ vanishes on $C$. However, it does not lie in the ideal generated by the two polynomials.

(c) The Segre embedding restricted to the affine open $A^1 \times A^1$ is
\[ ((1 : x), (1 : y)) \mapsto (1 : x : y : xy). \]
If we compose this with the map $\phi : A^1 \to A^1 \times A^1$,
\[ t \mapsto ((1 : t), (1 : t^2)) \]
then the image will be the twisted cubic.

The image of $\phi$ is defined by the equation $x^2 = y$. With homogeneous coordinates $x = X_1/X_0$ and $y = Y_1/Y_0$, the equation is $X_1^2Y_0 = X_0^2Y_1$. This is an equation of bidegree $(2, 1)$.

**PROBLEM 3.** Consider the Segre map $s_{1,2} : P^1 \times P^2 \to P^5$. The image of the map is the variety
\[ X = \{(Z_0 : \ldots : Z_5) \in P^5 | \text{rank} \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_3 & Z_4 & Z_5 \end{pmatrix} \leq 1 \}. \]

(a) Describe the second projection $\pi_2 : P^1 \times P^2 \to P^2$ in the coordinates $Z_i$. Show that it is regular everywhere.
(b) Prove that the twisted cubic $C \subset P^5$ is isomorphic to the intersection of $X$ with a linear 3-plane $P^3 \subset P^5$.

(a) The Segre map is
\[ (X_0 : X_1), (Y_0, Y_1, Y_2) \mapsto (Z_0 : Z_1 : \ldots : Z_5) = (X_0Y_0 : X_0Y_1 : X_1Y_2 : X_1Y_0 : X_0Y_1 : X_1Y_2). \]

The projection map is given by $(Z_0 : Z_1 : Z_2)$ or equivalently by $(Z_3 : Z_4 : Z_5)$.
These vectors don’t vanish simultaneously.

(b) The twisted cubic is the closure of the set
\[ \begin{pmatrix} 1 & t & t^2 \\ t & t^2 & t^3 \end{pmatrix}. \]
This set lies in $X$ and it is defined by linear equations $Z_3 = Z_1$ and $Z_4 = Z_2$.

**PROBLEM 4.** Given $m$ points in $P^n$, we can consider the $m$-tuple as a point in $P^n \times P^n \times \cdots \times P^n$. Generalizing the Segre embedding, we can map this product into $P^{(n+1)^m-1}$:
\[ ((X_0 : \ldots : X_n), (Y_0 : \ldots : Y_n), \ldots, (Z_0 : \ldots : Z_n)) \mapsto (X_0Y_0 \cdots Z_0 : \ldots : X_iY_j \cdots Z_l : \ldots : X_nY_n \cdots Z_n). \]

(a) Consider 3 points in $P^2$. Prove that the locus of three collinear points (lying on a line) is closed in $(P^2)^3$ by showing that this locus is the intersection of $(P^2)^3$ with a hyperplane in $P^{26}$.
(b) Repeat the previous part with 4 collinear points in $P^2$.

(a) Collinearity of three points is defined by the vanishing of the $3 \times 3$ determinant.
This is an equation of tri-degree $(1, 1, 1)$, hence it is given by one linear equation in $P^{26}$.
(b) We can put the coordinates of the 4 points in a $3 \times 4$ matrix. Then all 4 minors of this matrix must vanish. Each minor corresponds to a linear equation.
**Problem 5.** Let \( X \subset \mathbb{P}^n \) be a projective variety, not equal to a finite set of points. We proved in class that the only regular functions on \( X \) are constant functions.

(a) Let \( H \subset \mathbb{P}^n \) be a hyperplane, \( H = V(F) \), where \( F \) is linear. Prove that \( X \cap H \neq \emptyset \).

(b) Let \( H \subset \mathbb{P}^n \) be a hypersurface, \( H = V(F) \), where \( \deg F = d > 0 \). Prove that \( X \cap H \neq \emptyset \). (Hint: use the Veronese embedding of \( \mathbb{P}^n \).)

(c) Prove that any two curves in \( \mathbb{P}^2 \) intersect. A curve in \( \mathbb{P}^2 \) is a hypersurface \( V(F) \).

(a) If \( H = V(f) \) then \( g/f \) with \( g \) and \( f \) both linear would be regular on \( X \). We can choose \( g \) to vanish at some point of \( X \) and not at another, hence \( g/f \) is not a constant function.

(b) The same argument as in the previous part works with \( f, g \) of any degree.

(c) Clear from the previous part.

**Problem 6.** A plane cubic curve is \( C = V(f) \subset \mathbb{P}^2 \), where \( f \) is a degree 3 homogeneous polynomial. Note that \( f \) and \( cf \) for \( c \neq 0 \) define the same curve. Let \( W = k[X_0, X_1, X_2]_3 \) be the vector space of homogeneous degree 3 polynomials (and 0). Let \( \mathbb{P}(W) \) be the set of lines in \( W \), which we identify with the set of plane cubics. Choosing the 10 degree 3 monomials as a basis, we have \( W \cong k^{10} \) and \( \mathbb{P}(W) \cong \mathbb{P}^9 \).

Define the incidence variety

\[
Y = \{(P, C) \in \mathbb{P}^2 \times \mathbb{P}(W) | P \in C\},
\]

and let \( \pi_1, \pi_2 \) be the two projections from \( Y \) to \( \mathbb{P}^2 \) and \( \mathbb{P}(W) \).

(a) Describe the fibres of \( \pi_1 \) and \( \pi_2 \). Prove that

\[
\pi_1^{-1}(P) = \mathbb{P}(W_P)
\]

for some linear subspace \( W_P \subset W \) of codimension 1.

(b) Prove that, given any 9 points \( P_1, \ldots, P_9 \in \mathbb{P}^2 \), there exists a cubic through them.

(a) The fibre of \( \pi_2 \) over \( C \) is the curve \( C \). The fibre of \( \pi_1 \) over a point \( P \) is the set of all cubics passing through \( P \). This is a linear hyperplane in \( \mathbb{P}(W) \).

(b) Nine hyperplanes in \( k^{10} \) always have a non-zero intersection. This means that nine hyperplanes in \( \mathbb{P}(W) \) intersect at some point. This intersection point is a cubic curve passing through all nine points.