MATH 532 - HOMEWORK #4
Due Tuesday, Nov. 12

Problem 1. Let $X = Z(y^2 - g(x)) \subset \mathbb{A}^2$ where $g(x)$ is a polynomial of degree 3, and let $\pi : (x, y) \mapsto x$ be the projection. The projective closure $\overline{X} \subset \mathbb{P}^2$ is defined by the homogenization of the polynomial $y^2 - g(x)$. (Set $x = X_1/X_0, y = X_2/X_0$ and clear denominators in the rational function.)

1. Find all points at infinity $\overline{X} \setminus X$ and show that $\pi$ extends to a regular map $\overline{\pi} : \overline{X} \to \mathbb{P}^1$. (Hint: the map is $\overline{\pi} : (X_0 : X_1 : X_2) \mapsto (X_0 : X_1)$. You can change the right hand side to $(X_0X_1^2 : X_1^3)$ and simplify.)

2. Repeat the previous part with $X = Z(y^3 - g(x))$, with $g(x)$ as before.

Problem 2. Consider the twisted cubic $C = \{(t, t^2, t^3)|t \in k\} \subset \mathbb{A}^3$. Its closure $\overline{C} \subset \mathbb{P}^2$ is the image of the Veronese map $v_3 : \mathbb{P}^1 \to \mathbb{P}^3$.

(a) The curve $C$ can be defined by equations $y = x^2, z = x^3$. Prove that the homogenizations of these equations do not define the closure $\overline{C}$. Find another equation $E$ for $C$, so that the homogenizations of $E$ and the other two equations define $\overline{C}$.

(b) Show that $C = V(y-x^2, z^2-2xyz+y^3)$ and $\overline{C}$ is defined by the homogenizations of the two polynomials, but the ideal in $k[X_0, X_1, X_2]$ generated by the two homogenized polynomials is not radical.

(c) Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, defined by $Z_{01}Z_{10} = Z_{00}Z_{11}$. Prove that $\overline{C}$ lies on $\mathbb{P}^1 \times \mathbb{P}^1$ (with the correct choice of coordinates) and it can be defined by one bi-homogeneous polynomial. Determine the bi-degree of this polynomial.

Problem 3. Consider the Segre map $s_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$. The image of the map is the variety

$$X = \{(Z_0 : \ldots : Z_5) \in \mathbb{P}^5| \text{rank } \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_3 & Z_4 & Z_5 \end{pmatrix} \leq 1\}.$$

(a) Describe the second projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ in the coordinates $Z_i$. Show that it is regular everywhere.

(b) Prove that the twisted cubic $\overline{C} \subset \mathbb{P}^3$ is isomorphic to the intersection of $X$ with a linear 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$.

Problem 4. Given $m$ points in $\mathbb{P}^n$, we can consider the $m$-tuple as a point in $\mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n$. Generalizing the Segre embedding, we can map this product into $\mathbb{P}^{(n+1)m-1}$:

$$((X_0 : \ldots : X_n), (Y_0 : \ldots : Y_n), \ldots, (Z_0 : \ldots : Z_n)) \mapsto (X_0Y_0 \cdots Z_0 : \ldots : X_1Y_1 \cdots Z_1 : \ldots : X_nY_n \cdots Z_n).$$

1. Consider 3 points in $\mathbb{P}^2$. Prove that the locus of three colinear points (lying on a line) is closed in $(\mathbb{P}^2)^3$ by showing that this locus is the intersection of $(\mathbb{P}^2)^3$ with a hyperplane in $\mathbb{P}^{26}$.

2. Repeat the previous part with 4 colinear points in $\mathbb{P}^2$. 


**Problem 5.** Let $X \subset \mathbb{P}^n$ be a projective variety, not equal to a finite set of points. We proved in class that the only regular functions on $X$ are constant functions.

(a) Let $H \subset \mathbb{P}^n$ be a hyperplane, $H = V(F)$, where $F$ is linear. Prove that $X \cap H \neq \emptyset$.

(b) Let $H \subset \mathbb{P}^n$ be a hypersurface, $H = V(F)$, where $\deg F = d > 0$. Prove that $X \cap H \neq \emptyset$. (Hint: use the Veronese embedding of $\mathbb{P}^n$.)

(c) Prove that any two curves in $\mathbb{P}^2$ intersect. A curve in $\mathbb{P}^2$ is a hypersurface $V(F)$.

**Problem 6.** A plane cubic curve is $C = V(f) \subset \mathbb{P}^2$, where $f$ is a degree 3 homogeneous polynomial. Note that $f$ and $cf$ for $c \neq 0$ define the same curve. Let $W = k[X_0, X_1, X_2]_3$ be the vector space of homogeneous degree 3 polynomials (and 0). Let $\mathbb{P}(W)$ be the set of lines in $W$, which we identify with the set of plane cubics. Choosing the 10 degree 3 monomials as a basis, we have $W \simeq k^{10}$ and $\mathbb{P}(W) \simeq \mathbb{P}^9$. Define the incidence variety

$$Y = \{(P, C) \in \mathbb{P}^2 \times \mathbb{P}(W)| P \in C\},$$

and let $\pi_1, \pi_2$ be the two projections from $Y$ to $\mathbb{P}^2$ and $\mathbb{P}(W)$.

(1) describe the fibres of $\pi_1$ and $\pi_2$. Prove that

$$\pi_1^{-1}(P) = \mathbb{P}(W_P)$$

for some linear subspace $W_P \subset W$ of codimension 1.

(2) Prove that, given any 9 points $P_1, \ldots, P_9 \in \mathbb{P}^2$, there exists a cubic through them.