In algebraic geometry we often work over a base $S$. This means that we consider a (pre-)variety $X$ together with a morphism $f : X \to S$. A morphism $f$ can be thought of as a family of varieties. For each point $s \in S$ there is the fibre $f^{-1}(s)$ and these fibres vary algebraically.

This homework contains problems that generalize notions such as product, separated, complete from varieties to varieties over a base. Most problems here can be done abstractly using the universal properties.

Let $C$ be a category, for example the category of sets, topological spaces, prevarieties, varieties. Let $f : X \to S$ and $g : Y \to S$ be morphisms in the category. The fibre product of $X$ and $Y$ over $S$ is an object denoted $X \times_S Y$ with two morphisms $\pi_1, \pi_2$ to $X$ and $Y$, respectively, such that $f \circ \pi_1 = g \circ \pi_2$:

$$
\begin{array}{ccc}
X \times_S Y & \xrightarrow{\pi_2} & Y \\
\pi_1 \downarrow & & \downarrow g \\
X & \xrightarrow{f} & S.
\end{array}
$$

The fibre product must satisfy the universal property: Given any object $Z$ and morphisms $p_1 : Z \to X$ and $p_2 : Z \to Y$ such that $f \circ p_1 = g \circ p_2$, there exists a unique morphism $\phi : Z \to X \times_S Y$ such that $p_1 = \pi_1 \circ \phi$ and $p_2 = \pi_2 \circ \phi$. (Another way to state the universal property is that to give a morphism $Z \to X \times_S Y$ is the same as to give two morphisms $p_1$ and $p_2$, such that $f \circ p_1 = g \circ p_2$.) A fibre product is unique up to isomorphism if it exists.

**Problem 1.** Let $f : X \to S$ and $g : Y \to S$ be maps in the category of sets. The fibre product of sets exists and it is $X \times_S Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$.

Describe in more common terms (such as union, intersection, fibre, inverse image, product, etc) and without proof the fibre products in the following cases:

(a) $S$ is a one point set.
(b) $X$ is a one point set.
(c) $f$ is an inclusion.
(d) Both $f$ and $g$ be inclusions.

(a) product $X \times Y$.
(b) Fibre $g^{-1}(f(P))$.
(c) Inverse image $g^{-1}(X)$.
(d) Intersection $X \cap Y$.

**Problem 2.** Fibre products exist in the category of prevarieties. This problem proves it when $S$ is a variety. Let $f : X \to S$ and $g : Y \to S$ be morphisms of prevarieties. Assume $S$ is a variety.

(a) Show that the fibre product of sets $X \times_S Y$ is a Zariski closed subset of $X \times Y$, hence a sub-prevariety. (Hint: consider the morphism $X \times Y \to S \times S$.)

(b) Prove that the prevariety $X \times_S Y$ constructed above satisfies the universal property. (Use the universal property of the product. Note that a morphism
to a sub-prevariety is the same as a morphism to the ambient prevariety with image in the sub-prevariety.)
(c) Let \( X \rightarrow T \rightarrow S \) and \( Y \rightarrow S \) be morphisms. Prove that
\[
X \times_T (T \times_S Y) \cong X \times_S Y.
\]
(Hint: Show that the left hand side satisfies the universal property of \( X \times_S Y \).)

(a) The fibre product is the inverse image of the diagonal \( \Delta \) by the map \( X \times Y \rightarrow S \times S \). The diagonal is closed if \( S \) is separated.
(b) Given \( p_1 : Z \rightarrow X \) and \( p_2 : Z \rightarrow Y \) such that \( f \circ p_1 = g \circ p_2 \), there exists a unique morphism to \( X \times Y \) from the universal property of the product. We need to check that this morphism maps to the closed subvariety \( X \times_S Y \). This can be checked for every point \( z \in Z \).
(c) Let us check that the left hand side satisfies the universal property of \( X \times_S Y \).

Let \( p_1 : Z \rightarrow X \) and \( p_2 : Z \rightarrow Y \) such that \( f \circ p_1 = g \circ p_2 \). First we use the universal property of \( T \times_S Y \) applied to the morphisms \( Z \rightarrow X \rightarrow T \) and \( Z \rightarrow Y \) to get a unique morphism \( Z \rightarrow T \times_S Y \). Then we use the universal property of \( X \times_T (T \times_S Y) \) applied to this morphism and \( Z \rightarrow X \) to get a unique morphism \( Z \rightarrow X \times_T (T \times_S Y) \). One can easily check at each step that the appropriate compositions are equal.

A morphism \( \psi : X \rightarrow S \) of prevarieties is called \emph{separated} if, given any two morphisms \( f, g : Z \rightarrow X \), such that \( \psi \circ f = \psi \circ g \), the set
\[
Eq(f, g) = \{ z \in Z | f(z) = g(z) \}
\]
is closed in \( Z \). Notice that a prevariety \( X \) is separated if the map \( X \rightarrow \{ \text{pt} \} \) is separated.

**Problem 3.** Give short proofs of the following:
(a) Let \( X \) be the line with doubled origin and \( \psi : X \rightarrow \mathbb{A}^1 \) the map that is the identity on each of the two affine pieces of \( X \). Then \( \psi \) is not separated.
(b) If \( X \) is separated, then any morphism \( X \rightarrow S \) is separated.
(c) \( X \rightarrow S \) is separated if and only if the diagonal \( \Delta \subseteq X \times_S X \) is closed.
(d) Let \( Z \subset X \) be a sub-prevariety (open or closed). Then the inclusion map \( Z \rightarrow X \) is separated.
(e) The composition \( X \rightarrow Y \rightarrow Z \) of separated morphisms is separated.
(f) If \( \psi \circ \phi \) is separated, so is \( \phi \). (Note that (2) is a special case of this.)
(g) If \( X \rightarrow S \) is separated and \( Y \rightarrow S \) any morphism, then \( X \times_S Y \rightarrow Y \) is also separated. (Hint: use the universal property of fibre products to describe maps to \( X \times_S Y \).)

(a) Let \( f, g : \mathbb{A}^1 \rightarrow X \) be the inclusions of the two affine pieces. Then \( \psi \circ f = \psi \circ g \) and \( Eq(f, g) = \mathbb{A}^1 \setminus \{0\} \).
(b) The condition for a morphism to be separated is weaker than the condition that \( X \) is separated. For the morphism we don’t need to consider all pairs of morphisms, only those that compose to the same morphism to \( S \).
(c) The locus \( Eq(f, g) \) is the inverse image of the diagonal by the map \( Z \rightarrow X \times_S X \). If the diagonal is closed, then so is its inverse image. Conversely, if the diagonal is not closed, then the two projections \( f, g : Z = X \times_S X \rightarrow X \) have \( Eq(f, g) = \Delta \) that is not closed.
(d) If for two morphisms \( f, g : W \to Z \) their compositions with \( Z \to X \) are equal, then \( f \) and \( g \) are equal, hence \( Eq(f, g) = W \).

(e) Let \( f, g : W \to X \) be two morphisms that compose to the same morphism to \( Z \). Then using that \( Y \to Z \) is separated, there is a closed subset \( V \subset W \) where \( f \) and \( g \) compose to the same morphism to \( Y \). Now separatedness of \( X \to Y \) gives a closed subset of \( V \) on which \( f \) and \( g \) agree.

(f) If \( f \) and \( g \) become equal after composing with \( \phi \), then they stay equal after further composing with \( \psi \). This means that separatedness of \( \psi \circ \phi \) is a stronger condition as we need to consider more pairs of morphisms.

(g) A morphism to \( f : Z \to X \times_S Y \) is the same as a morphism to \( f_1 : Z \to X \) and a morphism to \( f_2 : Z \to Y \) that compose to the same morphism to \( S \). Given two such morphisms, \( f \) and \( g \) such that \( f_2 = g_2 \),

\[
Eq(f, g) = Eq(f_1, g_1).
\]

The right hand side is closed by separatedness of \( X \to S \). (Note that \( f_1 \) and \( g_1 \) compose to the same map to \( S \), which is the same map as the composition of \( f_2 = g_2 \) with \( Y \to S \).)

A morphism \( \psi : X \to S \) of prevarieties is called proper if it is separated and universally closed. This means that, given any \( W \to S \), the projection \( X \times_S W \to W \) is a closed map, taking closed sets to closed sets. Note that \( X \) is complete iff the map \( X \to \{pt\} \) is proper.

**Problem 4.** Prove:

(a) Let \( Z \subset X \) be a closed sub-prevariety. Then the inclusion map \( Z \to X \) is proper.

(b) If \( X \) is complete and \( S \) is separated, then any morphism \( X \to S \) is proper.  
   (Hint: \( X \times_S W \) is a closed sub-prevariety of \( X \times W \).)

(c) The composition \( X \to Y \to Z \) of proper morphisms is proper.

(d) If \( X \to S \) is proper and \( Y \to S \) any morphism, then \( X \times_S Y \to Y \) is also proper.

(a) If \( Z \to X \) is the inclusion of a closed subset, then \( W \times_X Z \to W \) is again the inclusion of a closed subset. This map is closed.

(b) The composition \( X \times_S W \to X \times W \to W \) is closed because each map is closed.

(c) Given \( W \to Z \), the morphism \( W \times_Z X \to W \) can be factored as

\[
W \times_Z X \to W \times_Z Y \to W.
\]

These two morphisms are closed by the properness of \( X \to Y \) and \( Y \to Z \).

(d) Given \( W \to Y \),

\[
W \times_Y (X \times_S Y) \simeq W \times_S X.
\]

The map from the right hand side to \( W \) is closed by the properness of \( X \to S \).