In algebraic geometry we often work over a base $S$. This means that we consider a (pre-)variety $X$ together with a morphism $f : X \to S$. A morphism $f$ can be thought of as a family of varieties. For each point $s \in S$ there is the fibre $f^{-1}(s)$ and these fibres vary algebraically.

This homework contains problems that generalize notions such as product, separated, complete from varieties to varieties over a base. Most problems here can be done abstractly using the universal properties.

Let $C$ be a category, for example the category of sets, topological spaces, prevarieties, varieties. Let $f : X \to S$ and $g : Y \to S$ be morphisms in the category. The fibre product of $X$ and $Y$ over $S$ is an object denoted $X \times_S Y$ with two morphisms $\pi_1, \pi_2$ to $X$ and $Y$, respectively, such that $f \circ \pi_1 = g \circ \pi_2$:

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{\pi_2} & Y \\
\pi_1 \downarrow & & \downarrow g \\
X & \xrightarrow{f} & S.
\end{array}
\]

The fibre product must satisfy the universal property: Given any object $Z$ and morphisms $p_1 : Z \to X$ and $p_2 : Z \to Y$ such that $f \circ p_1 = g \circ p_2$, there exists a unique morphism $\phi : Z \to X \times_S Y$ such that $p_1 = \pi_1 \circ \phi$ and $p_2 = \pi_2 \circ \phi$. (Another way to state the universal property is that to give a morphism $Z \to X \times_S Y$ is the same as to give two morphisms $p_1$ and $p_2$, such that $f \circ p_1 = g \circ p_2$.) A fibre product is unique up to isomorphism if it exists.

**Problem 1.** Let $f : X \to S$ and $g : Y \to S$ be maps in the category of sets. The fibre product of sets $X \times_S Y$ is a Zariski closed subset of $X \times Y$, hence a sub-prevariety. (Hint: consider the morphism $X \times Y \to S \times S$.) A fibre product is unique up to isomorphism if it exists.

**Problem 2.** Fibre products exist in the category of prevarieties. This problem proves it when $S$ is a variety.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of prevarieties. Assume $S$ is a variety.

(a) Show that the fibre product of sets $X \times_S Y$ is a Zariski closed subset of $X \times Y$, hence a sub-prevariety. (Hint: consider the morphism $X \times Y \to S \times S$.)

(b) Prove that the prevariety $X \times_S Y$ constructed above satisfies the universal property. (Use the universal property of the product. Note that a morphism to a sub-prevariety is the same as a morphism to the ambient prevariety with image in the sub-prevariety.)

(c) Let $X \to T \to S$ and $Y \to S$ be morphisms. Prove that

\[X \times_T (T \times_S Y) \simeq X \times_S Y.\]

(Hint: Show that the left hand side satisfies the universal property of $X \times_S Y$.)
A morphism $\psi : X \to S$ of prevarieties is called \textit{separated} if, given any two morphisms $f, g : Z \to X$, such that $\psi \circ f = \psi \circ g$, the set
\[ Eq(f, g) = \{ z \in Z | f(z) = g(z) \} \]
is closed in $Z$. Notice that a prevariety $X$ is separated iff the map $X \to \{ pt \}$ is separated.

**Problem 3.** Give short proofs of the following:
(a) Let $X$ be the line with doubled origin and $\psi : X \to \mathbb{A}^1$ the map that is the identity on each of the two affine pieces of $X$. Then $\psi$ is not separated.
(b) If $X$ is separated, then any morphism $X \to S$ is separated.
(c) $X \to S$ is separated if and only if the diagonal $\Delta \subseteq X \times_S X$ is closed.
(d) Let $Z \subset X$ be a sub-prevariety (open or closed). Then the inclusion map $Z \to X$ is separated.
(e) The composition $X \to Y \to Z$ of separated morphisms is separated.
(f) If $\psi \circ \phi$ is separated, so is $\phi$. (Note that (2) is a special case of this.)
(g) If $X \to S$ is separated and $Y \to S$ any morphism, then $X \times_S Y \to Y$ is also separated. (Hint: use the universal property of fibre products to describe maps to $X \times_S Y$.)

A morphism $\psi : X \to S$ of prevarieties is called \textit{proper} if it is separated and universally closed. This means that, given any $W \to S$, the projection $X \times_S W \to W$ is a closed map, taking closed sets to closed sets. Note that $X$ is complete iff the map $X \to \{ pt \}$ is proper.

**Problem 4.** Prove:
(a) Let $Z \subset X$ be a closed sub-prevariety. Then the inclusion map $Z \to X$ is proper.
(b) If $X$ is complete and $S$ is separated, then any morphism $X \to S$ is proper. (Hint: $X \times_S W$ is a closed sub-prevariety of $X \times W$.)
(c) The composition $X \to Y \to Z$ of proper morphisms is proper.
(d) If $X \to S$ is proper and $Y \to S$ any morphism, then $X \times_S Y \to Y$ is also proper.