Problem 1. Let $X \subset \mathbb{A}^3$ be the curve $\{(t^3, t^4, t^5) | t \in k\}$.

(a) The polynomials $xz - y^2, x^4 - y^3$ lie in the ideal $(X)$. Show that $V(xz - y^2, x^4 - y^3)$ is larger than $X$ and describe the other irreducible component(s). (Hint: you can slice $\mathbb{A}^3$ by setting $y$ equal to a constant and finding the traces of the two closed sets.)

(b) Is $V(xz - y^2, x^5 - z^3) = X$? If not, find the irreducible components.

We will assume that the field has characteristic 0. The first variety is a union of $X$ and the $z$-axis. The second one is the union of $X$ and the curve $\{(t^3, -t^4, t^5)\}$.

Problem 2. Consider the ring $R \subset k[x, y]$ consisting of all polynomials $f(x, y)$ that do not contain terms with monomials $y^i, i > 0$. (Another way to define it is as the set of polynomials $f(x, y)$, such that $f(0, y)$ is a constant.) Prove that the maximal ideal $m$ consisting of all polynomials with no constant term is not finitely generated. (Hint: consider monomials $xy^i$ for $i \geq 0$. Prove that no finitely generated ideal of $R$ can contain all these monomials.)

The monomial $xy^i$ is not divisible by any other monomial in the ring except the monomial 1. Suppose $f_1, \ldots, f_n$ generate the ideal. Then a linear combination $f_1 g_1 + \cdots + f_n g_n, \quad g_j \in R$ can contain a monomial $xy^i$ with nonzero coefficient only if the monomial occurs in some $f_j$. If $i$ is large then this is not possible.

Problem 3. Let $f : X \to Y$ be a continuous surjective map of topological spaces. If $X$ is irreducible, prove that $Y$ is also irreducible.

If $Y = C_1 \cup C_2$ is a union of proper closed subsets, then $X = f^{-1}(C_1) \cup f^{-1}(C_2)$ is also a union of proper closed subsets.

Problem 4. (a) A topological space $X$ is called quasicompact if every open cover

$$X = \bigcup_{i \in I} U_i$$

has a finite subcover: there exists a finite subset $\{i_1, \ldots, i_n\} \subset I$, such that

$$X = \bigcup_{j=1}^n U_{i_j}.$$

Prove that $\mathbb{A}^n$ with Zariski topology is quasi-compact.

(b) A base for the topology on $X$ is a collection of open sets $\{U_i\}_{i \in I}$, such that every open set $U \subset X$ is a union of some of the $U_i$. Prove that the distinguished opens

$$U_f = \mathbb{A}^n \setminus Z(f)$$

form a base for the the Zariski topology on $\mathbb{A}^n$. 

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The quasicompactness property in terms of closed sets is that if the intersection of closed sets $\cap_{i \in I} C_i$ in $X$ is empty, then a finite intersection is also empty. With Zariski topology, the sum of ideals $I(C_i)$ is the trivial ideal $(1)$. This means that the element 1 is a finite linear combination of elements $f_i$ from these ideals. Since only a finite set of ideals is involved, their sum is also $(1)$.

A closed set is defined by the vanishing of a finite set of polynomials $f_i$. The complement of this closed set is the union of $D(f_i)$.

**Problem 5.** Let $(S, \leq)$ be a partially ordered set. Define $C \subset S$ to be closed if $s \leq t$ and $s \in C$ imply $t \in C$.

(a) Prove that this defines a topology on $S$.
(b) Show that a one point set $\{s\} \subset S$ may not be closed, and describe its closure $\overline{\{s\}}$. (The closure of a set is the intersection of all closed sets containing it.)
(c) Prove that $\{s\}$ is irreducible.

The axioms of topology are easy to check. The closure of a one point set is $\overline{\{s\}} = \{t \geq s\}$.

It is irreducible since any closed subset that contain the point $s$ also contains the whole set.

**Problem 6.** Let a group $G$ act on $A^n$. A set $C \subset A^n$ is called $G$-closed if it is closed in the Zariski topology and $G$ maps $C$ to $C$.

(a) Prove that this defines a topology on $A^n$.
(b) Find an example of such action and a $G$-closed $C$ such that $C$ is irreducible in the topology defined, but reducible in the Zariski topology.
(c) Let $G = Gal(\mathbb{C}/\mathbb{R})$ that acts on $A^1_\mathbb{C} = \mathbb{C}$. (Recall that the Galois group $G$ is a two element group that acts on $\mathbb{C}$ by complex conjugation.) Prove that there is a bijection between minimal nonempty $G$-closed sets in $A^1_\mathbb{C}$ and maximal ideals in $\mathbb{R}[x]$.

The axioms are again easy to check. The Galois action as in the last part has a $G$-orbit $\{i, -i\}$. It is irreducible as a $G$-set but not as a set in Zariski topology. A minimal $G$-set with the Galois action is either $\{a\}$ where $a \in \mathbb{R}$ or $\{b, \overline{b}\}$ where $b$ is not real. Maximal ideals in $\mathbb{R}[x]$ correspond to monic irreducible polynomials. These polynomials are of the form $x - a$ and $(x - b)(x - \overline{b})$. 