The field $k$ is assumed everywhere to be algebraically closed.

**Problem 1.** Let $X \subset \mathbb{A}^3$ be the curve $\{(t^3, t^4, t^5) | t \in k\}$. You may assume that $X$ is a closed algebraic set.

(a) The polynomials $xz - y^2, x^4 - y^3$ lie in the ideal $(X)$. Show that $V(xz - y^2, x^4 - y^3) > X$ and describe the other irreducible component(s). (Hint: you can slice $\mathbb{A}^3$ by setting $y$ equal to a constant and finding the traces of the two closed sets.)

(b) Is $V(xz - y^2, x^5 - z^3) = X$? If not, find the irreducible components.

**Problem 2.** Consider the ring $R \subset k[x, y]$ consisting of all polynomials $f(x, y)$ that do not contain terms with monomials $y^i, i > 0$. (Another way to define it is as the set of polynomials $f(x, y)$, such that $f(0, y)$ is a constant.) Prove that the maximal ideal $\mathfrak{m}$ consisting of all polynomials with no constant term is not finitely generated. (Hint: consider monomials $xy^i$ for $i \geq 0$. Prove that no finitely generated ideal of $R$ can contain all these monomials.)

**Problem 3.** Let $f : X \to Y$ be a continuous surjective map of topological spaces. If $X$ is irreducible, prove that $Y$ is also irreducible.

**Problem 4.** (a) A topological space $X$ is called *quasicompact* if every open cover

$$X = \bigcup_{i \in I} U_i$$

has a finite subcover: there exists a finite subset $\{i_1, \ldots, i_n\} \subset I$, such that

$$X = \bigcup_{j=1}^n U_{i_j}.$$

Prove that $\mathbb{A}^n$ with Zariski topology is quasi-compact.

(b) A *base* for the topology on $X$ is a collection of open sets $\{U_i\}_{i \in I}$, such that every open set $U \subset X$ is a union of some of the $U_i$. Prove that the distinguished opens

$$U_f = \mathbb{A}^n \setminus Z(f)$$

form a base for the Zariski topology on $\mathbb{A}^n$.

**Problem 5.** Let $(S, \leq)$ be a partially ordered set. Define $C \subset S$ to be closed if $s \leq t$ and $s \in C$ imply $t \in C$.

(a) Prove that this defines a topology on $S$.

(b) Show that a one point set $\{s\} \subset S$ may not be closed, and describe its closure $\overline{\{s\}}$. (The closure of a set is the intersection of all closed sets containing it.)

(c) Prove that $\{s\}$ is irreducible.
Problem 6. Let a group $G$ act on $\mathbb{A}^n$. A set $C \subset \mathbb{A}^n$ is called $G$-closed if it is closed in the Zariski topology and $G$ maps $C$ to $C$.

(a) Prove that this defines a topology on $\mathbb{A}^n$.

(b) Find an example of such action and a $G$-closed $C$ such that $C$ is irreducible in the topology defined, but reducible in the Zariski topology.

(c) Let $G = Gal(\mathbb{C}/\mathbb{R})$ that acts on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$. (Recall that the Galois group $G$ is a two element group that acts on $\mathbb{C}$ by complex conjugation.) Prove that there is a bijection between minimal nonempty $G$-closed sets in $\mathbb{A}^1_{\mathbb{C}}$ and maximal ideals in $\mathbb{R}[x]$. 