Please choose 6 problems. All problems are worth the same number of marks.

All varieties are defined over an algebraically closed field $k$, which you may assume to have characteristic zero. In the last part about schemes the field $k$ is allowed to be arbitrary.

**Dimension of fibres and irreducibility.**

**Problem 1.** Let $X$ be an irreducible variety and $f : Y \rightarrow X$ a proper surjective morphism. Assume that all fibres of $f$ are irreducible and have the same dimension. Prove that $Y$ is irreducible. (Hint: prove that every component of $Y$ is a union of fibres of $f$. Then study the images of the components.)

**Matrices.** Write $G(m,n)$ for the Grassmannian of $m$-dimensional subspaces of $k^n$, and $G(m-1,n-1)$ for the same variety of $(m-1)$-planes in $\mathbb{P}^{n-1}$.

**Problem 2.** Let $M_n$ be the set of $n \times n$ matrices with entries in $k$. We identify $M_n$ with $A^{n^2}$. Let $V_r \subset M_n$ be the set of matrices of rank at most $r$. Let

$$I = \{([W], A) \in G(m,n) \times M_n | W \subset Ker(A)\}.$$

You may assume that $I$ is a closed subset of $G(m,n) \times M_n$.

(1) Considering the projection to $G(m,n)$, prove that $I$ is irreducible and find its dimension. (The projection map is not proper, but Problem 1 also applies here. We could replace $M_n$ with its projectivization; the incidence correspondence $I$ defined in that case is irreducible if and only if the $I$ defined above is irreducible.)

(2) Prove that $V_r$ is irreducible and find its dimension.

(3) Find the dimension of the set of symmetric $n \times n$ matrices of rank $\leq r$. (It may not be very obvious that the set of symmetric matrices containing a fixed $W$ in their kernel has a dimension that is independent of $W$. If we think of the symmetric matrix as a quadratic form (assuming $\text{char}(k) \neq 2$), then we are considering the space of quadratic forms on the quotient $k^n/W$.)

**Plane curves.**

**Problem 3.** A degree $d$ plane curve $C = Z(f)$ in $\mathbb{P}^2$ is defined by a degree $d$ homogeneous nonzero polynomial $f(X_0, X_1, X_2)$. Two curves $C_1 = Z(f)$ and $C_2 = Z(g)$ are equal if the polynomials $f$ and $g$ differ by a constant factor. We parametrize the set of all plane curves of degree $d$ by the projectivization $\mathbb{P}^N$ of the space of all degree $d$ homogeneous polynomials in 3 variables. Write $[C] \in \mathbb{P}^N$ for the point corresponding to the curve $C$.

A curve $C = V(f)$ is singular at a point $P \in C$ if all first order partial derivatives of $f$ vanish at $P$. Since $f$ is homogeneous, we have

$$\sum_i x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$
Hence, if all partials vanish at \( P \), so does \( f \).

Let \( D \subset \mathbb{P}^N \) be the locus of singular curves. \( D \) is called the discriminant locus. Also let \( S \subset \mathbb{P}^N \times \mathbb{P}^2 \) be the locus of singularities:

\[
S = \{([C], P) \mid C \text{ is singular at } P\}.
\]

1. Prove that \( S \) is closed in \( \mathbb{P}^N \times \mathbb{P}^2 \).
2. Show that the fibres of the projection \( S \to \mathbb{P}^2 \) are irreducible of the same dimension. Conclude that \( S \) is irreducible and find its codimension in \( \mathbb{P}^N \times \mathbb{P}^2 \).
   (Hint: the fibres do not depend on the point in \( \mathbb{P}^2 \). Choose a convenient point, for example \((1:0:0)\) and describe its fibre.)
3. The curve \( Z(X_1^d + X_2^d) \) has one singular point. (No need to prove this.) Use this to show that \( D \subset \mathbb{P}^N \) is an irreducible hypersurface.
4. Given two distinct curves of degree \( d \), \( C_1 = V(f) \), \( C_2 = V(g) \), prove that \( Z(af + bg) \) is a singular curve for some \( a, b \in k \).

Resolutions of singularities of plane curves.

The blowup of \( \mathbb{A}^2 \) at the point 0 is:

\[
\text{Bl}_0(\mathbb{A}^2) = \{(x_1, x_2, y_1, y_2) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x_1 y_2 = x_2 y_1\}.
\]

Projection to the first factor defines a morphism \( \pi : \text{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2 \). The blowup is covered by 2 charts \( U_1, U_2 \), each isomorphic to \( \mathbb{A}^2 \), and the map \( \pi \) to \( \mathbb{A}^2 \) in the two charts is given by

\[
(u_1, u_2) \mapsto (u_1, u_1 u_2), \quad (u_1, u_2) \mapsto (u_1 u_2, u_2).
\]

One can similarly blow up \( \mathbb{A}^2 \) at any point \( P \) by choosing coordinates with \( P = 0 \). More generally, one can blow up a point \( P \in U \subset X \), where \( U \simeq \mathbb{A}^2 \).

If \( C \subset \mathbb{A}^2 \) is a curve passing through 0, then the strict transform of \( C \) is \( C' \subset \text{Bl}_0(\mathbb{A}^2) \), such that \( \pi^{-1}(C) = C' \cup \pi^{-1}(0) \) and \( C' \) does not contain \( \pi^{-1}(C) \).

One can resolve the singularities of any plane curve by blowing up singular points and replacing the curve by its strict transform.

**Problem 4.** Resolve the singularities of the following plane curves. They may need several blowups to resolve all singularities.

1. \( C = Z(x^3 - y^5) \).
2. \( C = Z(x^5y + xy^2 - x^4 - y^4) \)

Resolution of surface singularities.

Similarly to the case of \( \mathbb{A}^2 \), one can define the blowup of \( \mathbb{A}^3 \) at 0. The blowups is covered by three charts \( U_i \), each isomorphic to \( \mathbb{A}^3 \). The projection maps \( U_i \to \mathbb{A}^3 \) are:

\[
(u_1, u_2, u_3) \mapsto (u_1, u_1 u_2, u_1 u_3), (u_1, u_2, u_3) \mapsto (u_1 u_2, u_2 u_3), (u_1, u_2, u_3) \mapsto (u_1 u_3, u_2 u_3, u_3).
\]

Given a surface \( Y \subset \mathbb{A}^3 \), the strict transform of \( Y \) in \( \text{Bl}_0(\mathbb{A}^3) \) is defined similarly to the case of curves.

We can sometimes resolve the singularities of a surface by blowing up singular points and taking the strict transform. Suppose \( Y \) has only one singular point \( P \). Then in the resolution \( f : Y' \to Y \) we can consider the inverse image \( f^{-1}(P) \), consisting of a finite union of irreducible curves. The resolution graph of \( f \) is constructed as
follows. Take one node for each component of $f^{-1}(P)$ and connect two nodes with an edge if the two components intersect.

**Problem 5.** Let $Y$ have the $A_k$ singularity:

$$Y = V(x^{k+1} + y^2 + z^2),$$

where $k ≥ 1$. Find the resolution of $Y$ by a sequence of blowups of points. Find the graph of the resolution. (Hint: Each blowup should introduce two new components to the fibre $f^{-1}(P)$.)

**Problem 6.** Let $Y$ have the $D_k$ singularity:

$$Y = V(x^{k-1} + xy^2 + z^2),$$

where $k ≥ 4$. Find the resolution of $Y$ when $k = 4$ by a sequence of blowups of points. Find the graph of the resolution. (The variety $Y$ has one singular point. After one blowup there will be several singular points.)

**Schemes.**

**Problem 7.** Let $X = \text{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, let $Y = \text{Spec} \mathbb{R}[x]$, and let $f : X \to Y$ be the projection to the $x$-axis (considering $X$ as a closed subscheme of $\mathbb{A}^2_{\mathbb{R}}$ and $Y$ the $x$-axis $\mathbb{A}^1_{\mathbb{R}}$). Describe the fibres of $f$ over closed points of $Y$. (Hint: Given a maximal ideal in $\mathbb{R}[x]$, such as $P = (x^2 + 1)$, the fibre over $P$ is the scheme $\text{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1, x^2 + 1)$. Find the points in the fibre and their residue fields.)

**Problem 8.** Let $p$ be a prime number and $X = \mathbb{A}^1_{\mathbb{F}_p}$. Let $X(\mathbb{F}_p^n)$ be the set of morphisms of schemes

$$\text{Spec}(\mathbb{F}_p^n) \to X.$$ Elements of $X(\mathbb{F}_p^n)$ are called $\mathbb{F}_p^n$-valued points of $X$.

1. Find the number of elements in $X(\mathbb{F}_p^n)$. (Hint: a morphism of affine schemes is the same as a homomorphism of rings.)

2. A closed point of $P \in X$ is a maximal ideal in $\mathbb{F}_p[x]$, generated by an irreducible monic polynomial $f(x)$. The residue field of the point is $\mathbb{F}_p[x]/(f(x)) \simeq \mathbb{F}_{p^m}$, where $m$ is the degree of $f$. Let $N_m$ be the number of all such points with residue field $\mathbb{F}_{p^m}$, equivalently, the number of monic irreducible polynomials of degree $m$. Find the number of elements in $X(\mathbb{F}_p^n)$ in terms of the numbers $N_m$. (Note that to give a morphism $\text{Spec} K \to X$ is the same as to give a point $P \in X$ and an embedding of fields $\kappa(P) \to K$. Such field embeddings can be counted using Galois theory.)

3. Explain how to compute $N_m$ from the number of elements in $X(\mathbb{F}_p^n)$. You don’t need to find the exact formula for $N_m$, which requires Möbius inversion. It is enough to explain how the numbers $N_m$ can be computed for $m = 1, 2, 3, \ldots$.

**Problem 9.** Let $(X, \mathcal{O}_X)$ be a ringed space, $G$ a group acting on $(X, \mathcal{O}_X)$. Then the quotient $X/G$ can again be given the structure of a ringed space. Let $X/G$ be the quotient space, that means, the set of $G$-orbits. This set is given the quotient topology where $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in $X$. Here $\pi : X \to X/G$ is the quotient map. The sheaf of rings on $X/G$ consists of $G$-invariant sections of $\mathcal{O}_X$:

$$\mathcal{O}_{X/G}(U) = \mathcal{O}_X(\pi^{-1}(U))^G.$$
Note that if $X$ is covered by $G$-invariant open sets $V_i$, then $X/G$ is covered by the quotients $V_i/G$.

(1) Let $X = \mathbb{A}^2_k \setminus \{(0,0)\}$, let $G = k^*$, and let $G$ act by

$$t \cdot (x,y) = (tx,ty), \quad t \in k^*, \quad (x,y) \in X.$$ 

Show that $X/G = \mathbb{P}^1_k$. (Hint: show that $X$ is covered by two $G$-invariant affines, isomorphic to $\mathbb{A}^1_k \times G$. Describe the quotients of these charts and how the quotients are glued in $X/G$.)

(2) Let $X$ and $G$ be as in the previous part, but let the action be

$$t \cdot (x,y) = (tx,t^{-1}y), \quad t \in k^*, \quad (x,y) \in X.$$ 

Show that $X/G$ is the line $\mathbb{A}^1_k$ with doubled origin.

**Problem 10.** Let $X$ be a variety over $k = \overline{k}$, and $x \in X$ a closed point. The Zariski tangent space of $X$ at $x$ is

$$T_{X,x} = (m_x/m_x^2)^*.$$ 

Here $m_x \in \mathcal{O}_{X,x}$ is the maximal ideal of $x$ and $(\cdot)^*$ denotes the dual $k$-vector space. Suppose $x$ lies in an affine chart $\text{Spec} R$. Then we can replace $m_x$ with the maximal ideal of $x$ in $R$ without changing the quotient $m_x/m_x^2$. For example, if $X = \text{Spec} k[x_1, \ldots, x_n]$, and $x$ is the origin, then $m_x/m_x^2$ has basis $x_1, \ldots, x_n$. The dual vector space has the dual basis $e_1, \ldots, e_n$.

(1) Prove that to give a vector $v \in T_{X,x}$ is the same as to give a morphism of $k$-schemes $\text{Spec} k[t]/(t^2) \to X$, with the unique point of $\text{Spec} k[t]/(t^2)$ mapping to $x$.

(2) Let $X$ be an affine variety. Let $T_X$ be the union of all tangent spaces $T_{X,x}$. This is the set of all morphisms of $k$-schemes $\text{Spec} k[t]/(t^2) \to X$. Show that this set has naturally a scheme structure. To do this, start with $X = \text{Spec} k[x_1, \ldots, x_n]$. Then a $k$-algebra homomorphism $k[x_1, \ldots, x_n] \to k[t]/(t^2)$ is given by $x_i \mapsto a_i + b_it$. This gives a point $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{A}^{2n} = T_X$. The projection map $T_X \to X$ is given by $(a_1, \ldots, b_n) \mapsto (a_1, \ldots, a_n)$. When $X = \text{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, show that the $a_i, b_i$ must satisfy some polynomial equations, which then define $T_X$ as a closed subscheme of $\mathbb{A}^{2n}$.

**Problem 11.** Let $k$ be an algebraically closed field of characteristic $p$. Consider two groups: the additive group $(k,+)$ and the multiplicative group $(k^*, \cdot)$. Let $F : a \to a^p$ be the Frobenius map. It defines a group homomorphism for both groups. The kernel of $F$ is the trivial subgroup in both cases.

The two groups can be given the structure of an algebraic variety. We write the additive group as $\mathbb{G}_a = \text{Spec} k[x]$ and the multiplicative group as $\mathbb{G}_m = \text{Spec} k[x, x^{-1}]$. The group multiplication and inverse maps are defined by morphisms of varieties:

$$\mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, \quad \iota : \mathbb{G}_a \to \mathbb{G}_a.$$ 

Similarly, the Frobenius map is defined as a morphism $F : \mathbb{G}_a \to \mathbb{G}_a$. The Frobenius is a group homomorphism in the sense that

$$F \circ \mu = \mu \circ (F, F) : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a.$$ 

(1) Let $K$ be the kernel of $F : \mathbb{G}_a \to \mathbb{G}_a$, that is, $K$ is the inverse image scheme of the closed point $0 \in \mathbb{G}_a$. The multiplication map $\mu$ and the inverse map $\iota$ restrict to give morphisms of schemes

$$\mu : K \times K \to K, \quad \iota : K \to K.$$
We say that these morphisms give $K$ the structure of a group scheme. Describe
the scheme $K$ and the two morphisms $\mu$ and $\iota$.

(2) Let $N$ be the kernel of $F : \mathbb{G}_m \to \mathbb{G}_m$. Again, describe $N$ and the induced
scheme structure. Note that $N$ and $K$ are isomorphic as schemes, but
the group operations are different.

**Problem 12.** Recall that a closed point in an affine scheme $\text{Spec } R$ is a maximal
ideal in $R$.

(1) Show that every affine scheme $X$ has a closed point. Equivalently, show that
every ring has a maximal ideal. (Use Zorn’s lemma. Check the upper bound
condition for ideals.)

(2) Prove that every quasicompact topological space has a smallest nonempty closed
subset. Smallest means that it does not contain a smaller closed nonempty
subset. (Use Zorn’s lemma. Why is quasi-compactness needed?)

(3) Prove that every quasicompact scheme contains a closed point. (A closed subset
of a scheme can be given a scheme structure, similar to the case of a closed
subset of a variety. In particular, it is covered by open affines.)

(4) Find a non-quasicompact topological space that has no minimal nonempty
closed set.