Please choose 6 problems. All problems are worth the same number of marks.

All varieties are defined over an algebraically closed field $k$, which you may assume to have characteristic zero. In the last part about schemes the field $k$ is allowed to be arbitrary.

Dimension of fibres and irreducibility.

**Problem 1.** Let $X$ be an irreducible variety and $f : Y \to X$ a proper surjective morphism. Assume that all fibres of $f$ are irreducible and have the same dimension. Prove that $Y$ is irreducible. (Hint: prove that every component of $Y$ is a union of fibres of $f$. Then study the images of the components.)

**Matrices.** Write $G(m,n)$ for the Grassmannian of $m$-dimensional subspaces of $k^n$, and $G(m-1, n-1)$ for the same variety of $(m-1)$-planes in $\mathbb{P}^{n-1}$.

**Problem 2.** Let $M_n$ be the set of $n \times n$ matrices with entries in $k$. We identify $M_n$ with $A^{n^2}$. Let $V_r \subset M_n$ be the set of matrices of rank at most $r$. Let

$$ I = \{ ([W], A) \in G(m,n) \times M_n | W \subset Ker(A) \}. $$

You may assume that $I$ is a closed subset of $G(m,n) \times M_n$.

(1) Considering the projection to $G(m,n)$, prove that $I$ is irreducible and find its dimension. (The projection map is not proper, but Problem 1 also applies here. We could replace $M_n$ with its projectivization; the incidence correspondence $I$ defined in that case is irreducible if and only if the $I$ defined above is irreducible.)

(2) Prove that $V_r$ is irreducible and find its dimension.

(3) Find the dimension of the set of symmetric $n \times n$ matrices of rank $\leq r$. (It may not be very obvious that the set of symmetric matrices containing a fixed $W$ in their kernel has a dimension that is independent of $W$. If we think of the symmetric matrix as a quadratic form (assuming $chark \neq 2$), then we are considering the space of quadratic forms on the quotient $k^n/W$.)

Plane curves.

**Problem 3.** A degree $d$ plane curve $C = Z(f)$ in $\mathbb{P}^2$ is defined by a degree $d$ homogeneous nonzero polynomial $f(X_0, X_1, X_2)$. Two curves $C_1 = Z(f)$ and $C_2 = Z(g)$ are equal if the polynomials $f$ and $g$ differ by a constant factor. We parametrize the set of all plane curves of degree $d$ by the projectivization $\mathbb{P}^N$ of the space of all degree $d$ homogeneous polynomials in 3 variables. Write $[C] \in \mathbb{P}^N$ for the point corresponding to the curve $C$.

A curve $C = V(f)$ is singular at a point $P \in C$ if all first order partial derivatives of $f$ vanish at $P$. Since $f$ is homogeneous, we have

$$ \sum_i x_i \frac{\partial f}{\partial x_i} = d \cdot f. $$
Hence, if all partials vanish at \( P \), so does \( f \).

Let \( D \subseteq \mathbb{P}^N \) be the locus of singular curves. \( D \) is called the *discriminant locus*. Also let \( S \subseteq \mathbb{P}^N \times \mathbb{P}^2 \) be the locus of singularities:

\[
S = \{([C], P) | C \text{ is singular at } P \}.
\]

1. Prove that \( S \) is closed in \( \mathbb{P}^N \times \mathbb{P}^2 \).
2. Show that the fibres of the projection \( S \to \mathbb{P}^2 \) are irreducible of the same dimension. Conclude that \( S \) is irreducible and find its codimension in \( \mathbb{P}^N \times \mathbb{P}^2 \).
   (Hint: the fibres do not depend on the point in \( \mathbb{P}^2 \). Choose a convenient point, for example \((1 : 0 : 0)\) and describe its fibre.)
3. The curve \( Z(X_1^d + X_2^d) \) has one singular point. (No need to prove this.) Use this to show that \( D \subseteq \mathbb{P}^N \) is an irreducible hypersurface.
4. Given two distinct curves of degree \( d \), \( C_1 = V(f) \), \( C_2 = V(g) \), prove that \( Z(af + bg) \) is a singular curve for some \( a, b \in k \).

**Resolutions of singularities of plane curves.**

The blowup of \( \mathbb{A}^2 \) at the point \( 0 \) is:

\[
Bl_0(\mathbb{A}^2) = \{(x_1, x_2, y_1, y_2) \in \mathbb{A}^2 \times \mathbb{P}^1 | x_1 y_2 = x_2 y_1 \}.
\]

Projection to the first factor defines a morphism \( \pi : Bl_0(\mathbb{A}^2) \to \mathbb{A}^2 \). The blowup is covered by 2 charts \( U_1, U_2 \), each isomorphic to \( \mathbb{A}^2 \), and the map \( \pi \) to \( \mathbb{A}^2 \) in the two charts is given by

\[
(u_1, u_2) \mapsto (u_1, u_1 u_2), \quad (u_1, u_2) \mapsto (u_1 u_2, u_2).
\]

One can similarly blow up \( \mathbb{A}^2 \) at any point \( P \) by choosing coordinates with \( P = 0 \). More generally, one can blow up a point \( P \subseteq U \subseteq X \), where \( U \cong \mathbb{A}^2 \).

If \( C \subseteq \mathbb{A}^2 \) is a curve passing through \( 0 \), then the strict transform of \( C \) is \( C' \subseteq Bl_0(\mathbb{A}^2) \), such that \( \pi^{-1}(C) = C' \cup \pi^{-1}(0) \) and \( C' \) does not contain \( \pi^{-1}(C) \).

One can resolve the singularities of any plane curve by blowing up singular points and replacing the curve by its strict transform.

**Problem 4.** Resolve the singularities of the following plane curves. They may need several blowups to resolve all singularities.

1. \( C = Z(x^3 - y^5) \).
2. \( C = Z(x^2 y + xy^2 - x^4 - y^4) \)

**Resolution of surface singularities.**

Similarly to the case of \( \mathbb{A}^2 \), one can define the blowup of \( \mathbb{A}^3 \) at \( 0 \). The blowups is covered by three charts \( U_i \), each isomorphic to \( \mathbb{A}^3 \). The projection maps \( U_i \to \mathbb{A}^3 \) are:

\[
(u_1, u_2, u_3) \mapsto (u_1, u_1 u_2, u_1 u_3), (u_1, u_2, u_3) \mapsto (u_1 u_2, u_2 u_3), (u_1, u_2, u_3) \mapsto (u_1 u_3, u_2 u_3, u_3).
\]

Given a surface \( Y \subseteq \mathbb{A}^3 \), the strict transform of \( Y \) in \( Bl_0(\mathbb{A}^3) \) is defined similarly to the case of curves.

We can sometimes resolve the singularities of a surface by blowing up singular points and taking the strict transform. Suppose \( Y \) has only one singular point \( P \). Then in the resolution \( f : Y' \to Y \) we can consider the inverse image \( f^{-1}(P) \), consisting of a finite union of irreducible curves. The resolution graph of \( f \) is constructed as
follows. Take one node for each component of $f^{-1}(P)$ and connect two nodes with an edge if the two components intersect.

**Problem 5.** Let $Y$ have the $A_k$ singularity:

$$Y = V(x^{k+1} + y^2 + z^2),$$

where $k \geq 1$. Find the resolution of $Y$ by a sequence of blowups of points. Find the graph of the resolution. (Hint: Each blowup should introduce two new components to the fibre $f^{-1}(P)$.)

**Problem 6.** Let $Y$ have the $D_k$ singularity:

$$Y = V(x^{k-1} + xy^2 + z^2),$$

where $k \geq 4$. Find the resolution of $Y$ when $k = 4$ by a sequence of blowups of points. Find the graph of the resolution. (The variety $Y$ has one singular point. After one blowup there will be several singular points.)

**Schemes.**

**Problem 7.** Let $X = \text{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1)$, let $Y = \text{Spec} \mathbb{R}[x]$, and let $f : X \to Y$ be the projection to the $x$-axis (considering $X$ as a closed subscheme of $\mathbb{A}^2_\mathbb{R}$ and $Y$ the $x$-axis $\mathbb{A}^1_\mathbb{R}$). Describe the fibres of $f$ over closed points of $Y$. (Hint: Given a maximal ideal in $\mathbb{R}[x]$, such as $P = (x^2 + 1)$, the fibre over $P$ is the scheme $\text{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1, x^2 + 1)$. Find the points in the fibre and their residue fields.)

**Problem 8.** Let $p$ be a prime number and $X = \mathbb{A}^1_{\mathbb{F}_p}$. Let $X(\mathbb{F}_{p^n})$ be the set of morphisms of schemes

$$\text{Spec}(\mathbb{F}_{p^n}) \to X.$$

Elements of $X(\mathbb{F}_{p^n})$ are called $\mathbb{F}_{p^n}$-valued points of $X$.

1. Find the number of elements in $X(\mathbb{F}_{p^n})$. (Hint: a morphism of affine schemes is the same as a homomorphism of rings.)

2. A closed point of $P \in X$ is a maximal ideal in $\mathbb{F}_p[x]$, generated by an irreducible monic polynomial $f(x)$. The residue field of the point is $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^m}$, where $m$ is the degree of $f$. Let $N_m$ be the number of all such points with residue field $\mathbb{F}_{p^m}$, equivalently, the number of monic irreducible polynomials of degree $m$. Find the number of elements in $X(\mathbb{F}_{p^n})$ in terms of the numbers $N_m$. (Note that to give a morphism $\text{Spec} \mathbb{K} \to X$ is the same as to give a point $P \in X$ and an embedding of fields $\kappa(P) \to \mathbb{K}$. Such field embeddings can be counted using Galois theory.)

3. Explain how to compute $N_m$ from the number of elements in $X(\mathbb{F}_{p^n})$. You don’t need to find the exact formula for $N_m$, which requires Möbius inversion. It is enough to explain how the numbers $N_m$ can be computed for $m = 1, 2, 3, \ldots$.

**Problem 9.** Let $(X, \mathcal{O}_X)$ be a ringed space, $G$ a group acting on $(X, \mathcal{O}_X)$. Then the quotient $X/G$ can again be given the structure of a ringed space. Let $X/G$ be the quotient space, that means, the set of $G$-orbits. This set is given the quotient topology where $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in $X$. Here $\pi : X \to X/G$ is the quotient map. The sheaf of rings on $X/G$ consists of $G$-invariant sections of $\mathcal{O}_X$:

$$\mathcal{O}_{X/G}(U) = \mathcal{O}_X(\pi^{-1}(U))^G.$$
Note that if \( X \) is covered by \( G \)-invariant open sets \( V_i \), then \( X/G \) is covered by the quotients \( V_i/G \).

(1) Let \( X = \mathbb{A}^2_k \setminus \{(0,0)\} \), let \( G = k^\ast \), and let \( G \) act by
\[
t \cdot (x, y) = (tx, ty), \quad t \in k^\ast, \quad (x, y) \in X.
\]
Show that \( X/G = \mathbb{P}^1_k \). (Hint: show that \( X \) is covered by two \( G \)-invariant affines, isomorphic to \( \mathbb{A}^1_k \times G \). Describe the quotients of these charts and how the quotients are glued in \( X/G \).)

(2) Let \( X \) and \( G \) be as in the previous part, but let the action be
\[
t \cdot (x, y) = (tx, t^{-1}y), \quad t \in k^\ast, \quad (x, y) \in X.
\]
Show that \( X/G \) is the line \( \mathbb{A}^1_k \) with doubled origin.

**Problem 10.** Let \( X \) be a variety over \( k = \bar{k} \), and \( x \in X \) a closed point. The Zariski tangent space of \( X \) at \( x \) is
\[
T_{X,x} = (m_x/m_x^2)^\ast.
\]
Here \( m_x \in \mathcal{O}_{X,x} \) is the maximal ideal of \( x \) and \((\cdot)^\ast\) denotes the dual \( k \)-vector space. Suppose \( x \) lies in an affine chart \( \text{Spec} \, R \). Then we can replace \( m_x \) with the maximal ideal of \( x \) in \( R \) without changing the quotient \( m_x/m_x^2 \). For example, if \( X = \text{Spec} \, k[x_1, \ldots, x_n] \), and \( x \) is the origin, then \( m_x/m_x^2 \) has basis \( x_1, \ldots, x_n \). The dual vector space has the dual basis \( e_1, \ldots, e_n \).

(1) Prove that to give a vector \( v \in T_{X,x} \) is the same as to give a morphism of \( k \)-schemes \( \text{Spec} \, k[t]/(t^2) \rightarrow X \), with the unique point of \( \text{Spec} \, k[t]/(t^2) \) mapping to \( x \).

(2) Let \( X \) be an affine variety. Let \( T_X \) be the union of all tangent spaces \( T_{X,x} \).
This is the set of all morphisms of \( k \)-schemes \( \text{Spec} \, k[t]/(t^2) \rightarrow X \). Show that this set has naturally a scheme structure. To do this, start with \( X = \text{Spec} \, k[x_1, \ldots, x_n] \). Then a \( k \)-algebra homomorphism \( k[x_1, \ldots, x_n] \rightarrow k[t]/(t^2) \) is given by \( x_i \mapsto a_i + b_it \). This gives a point \( (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{A}^{2n} = T_X \).

The projection map \( T_X \rightarrow X \) is given by \( (a_1, \ldots, b_n) \mapsto (a_1, \ldots, a_n) \). When \( X = \text{Spec} \, k[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \), show that the \( a_i, b_i \) must satisfy some polynomial equations, which then define \( T_X \) as a closed subscheme of \( \mathbb{A}^{2n} \).

**Problem 11.** Let \( k \) be an algebraically closed field of characteristic \( p \). Consider two groups: the additive group \( (k, +) \) and the multiplicative group \( (k^\ast, \cdot) \). Let \( F : a \rightarrow a^p \) be the Frobenius map. It defines a group homomorphism for both groups. The kernel of \( F \) is the trivial subgroup in both cases.

The two groups can be given the structure of an algebraic variety. We write the additive group as \( \mathbb{G}_a = \text{Spec} \, k[x] \) and the multiplicative group as \( \mathbb{G}_m = \text{Spec} \, k[x, x^{-1}] \).

The group multiplication and inverse maps are defined by morphisms of varieties:
\[
\mu : \mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_a, \quad \iota : \mathbb{G}_a \rightarrow \mathbb{G}_a.
\]
Similarly, the Frobenius map is defined as a morphism \( F : \mathbb{G}_a \rightarrow \mathbb{G}_a \). The Frobenius is a group homomorphism in the sense that
\[
F \circ \mu = \mu \circ (F, F) : \mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_a.
\]

(1) Let \( K \) be the kernel of \( F : \mathbb{G}_a \rightarrow \mathbb{G}_a \), that is, \( K \) is the inverse image scheme of the closed point \( 0 \in \mathbb{G}_a \). The multiplication map \( \mu \) and the inverse map \( \iota \) restrict to give morphisms of schemes
\[
\mu : K \times K \rightarrow K, \quad \iota : K \rightarrow K.
\]
We say that these morphisms give \( K \) the structure of a group scheme. Describe the scheme \( K \) and the two morphisms \( \mu \) and \( \iota \).

(2) Let \( N \) be the kernel of \( F : \mathbb{G}_m \to \mathbb{G}_m \). Again, describe \( N \) and the induced group scheme structure. Note that \( N \) and \( K \) are isomorphic as schemes, but the group operations are different.

**Problem 12.** Recall that a closed point in an affine scheme \( \text{Spec} R \) is a maximal ideal in \( R \).

(1) Show that every affine scheme \( X \) has a closed point. Equivalently, show that every ring has a maximal ideal. (Use Zorn’s lemma. Check the upper bound condition for ideals.)

(2) Prove that every quasicompact topological space has a smallest nonempty closed subset. Smallest means that it does not contain a smaller closed nonempty subset. (Use Zorn’s lemma. Why is quasi-compactness needed?)

(3) Prove that every quasicompact scheme contains a closed point. (A closed subset of a scheme can be given a scheme structure, similar to the case of a closed subset of a variety. In particular, it is covered by open affines.)

(4) Find a non-quasicompact topological space that has no minimal nonempty closed set.