Problem 1. Find the Galois group of the polynomial $x^3 - 10$ over $\mathbb{Q}, \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-3})$. (Hint: use the discriminant.)

Problem 2. Let $E$ be the splitting field of $x^4 - 2$ over $\mathbb{Q}$. Find the lattice of intermediate fields of the extension $\mathbb{Q} \subset E$. Indicate the Galois correspondence between the intermediate fields and subgroups of the Galois group. Which subfields are Galois over $\mathbb{Q}$? (To check your computations, the Galois group should be isomorphic to $D_8$. The four roots of $x^4 - 2$ form the vertices of a square in $\mathbb{C}$ and the Galois group permutes the vertices by symmetries of the square.)

Problem 3. Consider the series

$$\chi(t) = t - t^5 + 10 \frac{t^9}{2!} - 15 \cdot 14 \frac{t^{13}}{3!} + 20 \cdot 19 \cdot 18 \frac{t^{17}}{4!} - \ldots.$$ 

Prove that $\chi(a)$ is a solution of the equation

$$x^5 + x - a = 0.$$ 

Substitute the power series into the equation and check the coefficients of $a, a^5, a^9, a^{13}$.

(This power series appeared as a footnote in an article by Eisenstein in 1844. It can be shown that all degree 5 equations in characteristic zero can be reduced using radicals to the one above, hence radicals and $\chi(t)$ can be used to solve all quintic equations.)