MATH 422/501 - HOMEWORK #4

Due Friday, Oct. 5

Problem 1. Factor the following polynomials into irreducibles:
(1) $x^3 + 6x^2 + 5x + 25$ in $\mathbb{F}_3[x]$, $\mathbb{Z}[x]$, $\mathbb{Q}[x]$.
(2) $x^4 + 8x + 12$ in $\mathbb{F}_5$, $\mathbb{Q}[x]$.
(3) $x^5 + x + 1$ in $\mathbb{F}_2[x]$.

Problem 2. (1) Consider $f(x) = x^n + a$ for some $n > 1$ and $a \in \mathbb{Z}$. For which $a$ does Eisenstein criterion prove that $f(x)$ is irreducible in $\mathbb{Z}[x]$? If $n = p$ is prime, for which $a$ does the Eisenstein criterion apply to the polynomial $f(x + b)$ with the given $p$.
(2) Consider a polynomial $f(x, y) \in F[x, y]$ for some field $F$. We think of a monomial $x^a y^b$ as a point $(a, b) \in \mathbb{R}^2$. The Newton polytope of $f(x, y)$ is the convex hull of all points $(a, b)$ such that $x^a y^b$ appears with nonzero coefficient in $f(x, y)$. Consider $f(x, y) \in R[y]$, where $R = F[x]$. When does Eisenstein criterion apply to $f(x, y)$ with prime $p = x$? State the condition in terms of the Newton polytope of $f(x, y)$.

Problem 3. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(1 + \sqrt[3]{2} + \sqrt[3]{4})$ where the cube roots are real numbers. The following parts can be done in any order:
(1) Find the degree of the extension.
(2) Find the minimal polynomial of the generator $1 + \sqrt[3]{2} + \sqrt[3]{4}$. (Hint: recall how we proved that a finite extension is algebraic.)