**MATH 422/501 - HOMEWORK #3**

*Due Friday, Sept. 28*

**Problem 1.** Let $R$ be the ring of continuous functions $f : [0, 10] \rightarrow \mathbb{R}$ and let $I_p \subset R$ be the ideal of functions vanishing at a point $p \in [0, 10]$.

1. Show that $R$ is not a domain. (Sketch some zero divisors.)
2. Prove that $I_p$ is a maximal ideal.
3. Show that every maximal ideal of $R$ is equal to $I_p$ for some $p$. (Hint: if an ideal is not contained in any $I_p$, show that it contains a unit and hence is equal to $R$.)
4. In the ring $S$ of all continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ find a proper ideal that is not contained in $I_p$ for any $p \in \mathbb{R}$. (Hint: think of functions vanishing at $p = \infty$.)

**Problem 2.** Let $R = \mathbb{R}[[x]]$ be the ring of formal power series in one variable. The order $\text{ord}(f)$ of a nonzero power series $f(x)$ is the smallest $n$ such that $f(x)$ has a nonzero term $a_n x^n$. The element 0 has order $\infty$.

1. Find the units in $R$. Use the order to describe this set. Show that every $f \in R$ can be expressed as $f = x^n u$ for some $n \geq 0$ and $u$ a unit.
2. Prove that $R$ is a domain.
3. Show that $R$ is a PID. (In fact, it is an Euclidean domain. There is a division algorithm with the order function describing the size of an element.) Describe the lattice of ideals in $R$.
4. Let $L = \mathbb{R}((x))$ be the ring of Laurent series

$$L = \left\{ \sum_{i \geq N} a_i x^i \mid N \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$ 

You may assume that $L$ is a ring. Prove that $L$ is the field of fractions of $R$.

**Problem 3.** Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. You may assume that this subset of $\mathbb{R}$ is in fact a subring of $\mathbb{R}$.

1. Show that $R$ is a PID. (In fact, $R$ is a Euclidean domain. Construct a division algorithm where the size of an element is measured by its norm:

$$N(a + b\sqrt{2}) = |(a + b\sqrt{2})(a - b\sqrt{2})| = |a^2 - 2b^2|.$$ 

To find the quotient in the division algorithm, first find the quotient in $\mathbb{R}$ and then take the “closest” element in $R$ to make the remainder small. Note: we are slightly abusing terminology here. As we will see later, the correct way to define the norm is as in the formula above but without taking the absolute value.)

2. Prove that $(3 - \sqrt{2})(3 + \sqrt{2})$ is a factorization of 7 into irreducibles. (Hint: show that the norm function is multiplicative and elements of norm 1 are units.)

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