MATH 422/501 - HOMEWORK #2

Due Friday, Sept. 21

Problem 1. Show that $S_n$ is a subgroup of $A_{n+2}$.

Problem 2. (a) Let $G$ be a simple group that has a subgroup of index $n$. Then $|G|$ divides $n!$
(b) Show that $A_6$ has no subgroup of prime index.

Problem 3. Show that the 5-cycles of $A_5$ fall into two conjugacy classes with 12 elements each. (Hint: Find the order of the centralizer of a 5-cycle in $S_5$. What is the centralizer of the same 5-cycle in $A_5$?)

Problem 4. Prove that $Z(S_4)$ is trivial and hence $S_4$ is isomorphic to its group of inner automorphisms. (One way to do this is as follows. The center is the intersection of the centralizers of all element. Compute the orders of some centralizers.)

Problem 5. Let $G$ be a group. A $G$-set is a set $A$ with the action of $G$. A homomorphism of $G$-sets is a map $f : A \to B$ such that $f(g \cdot a) = g \cdot f(a)$ for any $a, g$. A homomorphism is an isomorphism if it has an inverse, equivalently, if it is a bijection.

(a) Let $A, B$ be $G$-sets where the action on $A$ is transitive, $A = Ga$ for $a \in A$. Notice that a homomorphism of $G$-sets $f : A \to B$ is determined if we know $f(a)$ because $f(g \cdot a) = gf(a)$.
(i) If $f : A \to B$ is a homomorphism of $G$-sets, show that $\text{Stab}(a) \subseteq \text{Stab}(f(a))$.
(ii) Conversely, given $b \in B$ such that $\text{Stab}(a) \subseteq \text{Stab}(b)$, show that there exists a homomorphism $f$ such that $f(a) = b$. (The problem here is to prove that your $f$ is well-defined: if $ga = ha$, then $f(ga) = f(ha)$, and that the map $f$ is compatible with the group action.)

(b) For a subgroup $H \leq G$, let $G/H$ be the set of left cosets, $G/H = \{gH\}$. Let $G$ act on $G/H$ by left translation: $g_1 \cdot g_2 H = g_1 g_2 H$. Prove that two such $G$-sets $G/H$ and $G/K$ are isomorphic if and only if the subgroups $H$ and $K$ are conjugate.

(c) If $A$ is a $G$-set with transitive action, then $A$ is isomorphic to some $G/H$.
(d) Every $G$-set is isomorphic to a disjoint union of $G$-sets of the form $G/H$. 