Exam rules:

- You may refer to any result that was proved in class or that appeared in the homework. Any other nontrivial statement must be given a proof. If in doubt about using some result, please ask during the exam.
- There are 5 problems in this exam. The first 4 problems are worth 5 marks each, the last one 6 marks.
PROBLEM 1. Let $F$ be a field. Prove that a nonzero ideal $I \subset F[x]$ is contained in only a finite number of maximal ideals of $F[x]$. Describe these maximal ideals.

$\mathbb{F}_{[x]}$ is a PID:
\[ I = (f) . \]

$\mathbb{F}_{[x]}$ is also a UFD. Factor $f$ into irreducibles:
\[ f = g_1 \cdot g_2 \cdots g_n . \]

Then $(g_i)$ are maximal ideals containing $I$. 
Problem 2. Let $F \subset F(\alpha)$ be a field extension of odd degree. Prove that $F(\alpha^2) = F(\alpha)$.

\[
\begin{pmatrix}
F(\alpha^1) \\
F(\alpha^2) \\
F
\end{pmatrix}
\]

degree $1$ or $2$ because $\alpha^2 \in F(\alpha^1)$. $d$ odd

Since $2 \nmid d$, $\deg F(\alpha^1)/F(\alpha^2) = 1$, $F(\alpha) = F(\alpha^1)$. 
PROBLEM 3. Let $F$ be a finite field, $f(x) \in F[x]$ an irreducible polynomial, and $E$ the splitting field of $f(x)$. If $\alpha \in E$ is a root of $f(x)$, prove that $E = F(\alpha)$. (Hint: Is $F(\alpha)$ Galois over $F$?)

$F(\alpha)$ is a finite field, hence Galois over $\mathbb{F}_p$ for some $p$, and also Galois over $F$.

\[
\begin{array}{c}
F(\alpha) \\
\downarrow \\
F \\
\downarrow \\
\mathbb{F}_p
\end{array}
\quad \text{Galois} \quad \Rightarrow \quad \begin{array}{c}
F(\alpha) \\
\downarrow \\
F \\
\downarrow \\
\mathbb{F}_p
\end{array} \text{ Galois}.
\]

Since $F(\alpha)/F$ is Galois, it contains all conjugates of $\alpha$ over $F$. This means that all roots of $f(x)$ lie in $F(\alpha)$, $F(\alpha) = E$. 

Problem 4. Prove that the polynomial 

\[ f(x) = x^4 + 2x^2 + 4x + 3 \]

is irreducible in \( \mathbb{Q}[x] \). (Hint: reduce mod 3.)

\[ f(x) = x^4 + 2x^2 + x \pmod{3} \]

\[ = x(x^3 + 2x + 1) \]

The factor \( x^3 + 2x + 1 \) has no roots in \( \mathbb{Z}/3\mathbb{Z} \), hence it is irreducible in \( \mathbb{Z}/3\mathbb{Z}[x] \).

If \( f(x) \) factors in \( \mathbb{Q}[x] \) then it factors in \( \mathbb{Z}[x] \), and the factors reduce to \( x \) and \( x^3 + 2x + 1 \pmod{3} \).

Let's check that \( f(x) \) has no linear factor in \( \mathbb{Z}[x] \). A root of \( f(x) \) in \( \mathbb{Z} \) must divide 3, and must be equal to 0 mod 3. This gives two possible roots: 3, -3. Neither one is a root:

\[ 3^3 + 2 \cdot 3^2 + 4 \cdot 3 + 3 = 3 \left( 3^3 + 2 \cdot 3 + 4 + 3 \right) = 0 \]

\[ \Rightarrow 3 \cdot 2 \cdot 3^2 + 4 \cdot 3 = 0 \]

\[ \Rightarrow 4 = 0 \pmod{3}. \]
Problem 5. Let \( p \neq 2 \) be a prime and let \( \xi \in \mathbb{C} \) be a primitive \( p \)-th root of 1. 
(1) Show that
\[ \mathbb{Q}(\xi + \xi^{-1}) \subseteq \mathbb{Q}(\xi) \]
is an extension of degree 2.
(2) Show that \( \mathbb{Q} \subseteq \mathbb{Q}(\xi + \xi^{-1}) \) is a Galois extension and find its Galois group.

1) Note that
\[ f^* (f + f^{-1}) = f^2 + 1 \]
Hence \( f \) is a root of
\[ f(x) = x^2 - (f + f^{-1})x + 1, \]
and the extension has degree \( \leq 2 \).
The degree is not 1 because \( f + f^{-1} \in 1\mathbb{R} \) but \( f \notin 1\mathbb{R} \).

2) Galois correspondence
\[
\begin{array}{cccc}
\mathbb{Q} & \mathbb{Q}(f) & \mathbb{Q}(f + f^{-1}) & \mathbb{Q} \\
2 & 1 & H & G
\end{array}
\]
Here \( G = C_{p-1} \), \( H \) is a subgroup of order 2.
Since \( G \) is abelian, \( H \leq G \) is normal. Then
\[ \mathbb{Q}(f + f^{-1})/\mathbb{Q} \]
is Galois with Galois group
\[ C_{p-1}/C_2 \cong C_{p-1}/2 \].