PROBLEM 1. (a) Prove that for any real $\alpha$ we have for $x$ in $(-1,1)$
\[
(1 + x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n,
\]
where
\[
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.
\]
(Here one defines $x^\alpha = e^{\alpha \log x}$.)

You may assume that the Maclaurin series of $\log(1 + x)$ converges to $\log(1 + x)$ in $(-1,1)$, hence the the Maclaurin series of $(1 + x)^{\alpha}$ converges to $(1 + x)^{\alpha}$ in the same interval. Prove that the formula above is the Maclaurin series expansion.

A slightly roundabout way of proving the equality is to check that both sides satisfy the differential equation $y' = \frac{\alpha y}{1 + x}$, $y(0) = 1$, and apply Picard-Lidel"of theorem.)

(b) Prove that the Maclaurin series of $\sqrt{1 + x}$ is
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{2^{2n}(n!)^2(2n - 1)} x^n.
\]

Find the radius of convergence of this series. You can use the Stirling’s formula to apply the root test:
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.
\]

(c) Does the series in the previous part converge at $x = -1$? If it does, what is the limit?

PROBLEM 2. Power series can be used to solve differential equations. Consider the initial value problem
\[
(1 - x^2)y'''(x) - 2xy'(x) + 2y(x) = 0, \quad y(0) = 1, y'(0) = 0.
\]

Find a solution in the form of a power series $y(x) = \sum a_n x^n$ as follows.

(a) The differential equation gives a relation among coefficients $a_n$. Express $a_n$ in terms of $a_k$ for $k < n$. (Hint: substitute the series $y(x)$ into the differential equation to get that a series equals zero. This implies that all coefficients in the the new series must be zero.)

(b) Find a closed form expression for $a_n$.

(c) Express $y(x)$ in terms of elementary functions. (Hint: you can multiply with $x$, differentiation, integration of the power series to bring it to a simpler form.)
Problem 3. Formal power series are used as generating series in various counting problems. Here is one example. We consider finite sequences of parentheses that are well-formed in the sense that every opening parenthesis is followed by a matching closing parenthesis and vice versa. For example, the sequence 
\[
(())
\]
is allowed, but 
\[
())(()
\]
is not. Let \(c_n\) be the number of such sequences that contain \(n\) opening parentheses. Then
\[
\begin{align*}
  c_0 &= 1, \\
  c_1 &= 1, \quad (()) \\
  c_2 &= 2, \quad (())((),()) \\
  c_3 &= 5, \quad \ldots
\end{align*}
\]
Define the generating series
\[
F(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

(a) Call a sequence a wrap if it has the form \((s)\), where \(s\) is any allowable sequence. Prove that the generating function \(\sum d_n x^n\), where \(d_n\) counts the number of wraps with \(n\) opening parentheses is equal to \(xF(x)\). (For example, \(d_0 = 0\) as there are no wraps with 0 parentheses, \(d_1 = 1\) counting the wrap \((\), and so on.)

(b) Note that every sequence of length \(\geq 1\) can be written in a unique way as a concatenation \(s = s_1 s_2 \ldots s_m\), where \(s_i\) are wraps. For example, if \(s = (())())(()())\), then \(s_1 = (())\), \(s_2 = ()\), \(s_3 = (())\). Prove that
\[
F = 1 + xF + (xF)^2 + (xF)^3 + \cdots.
\]
To prove this, find what is the meaning of the coefficient of \(x^n\) in each summand \((xF)^2\), \((xF)^3\), \ldots.

(c) Solve for \(F\) in terms of elementary functions. (This involves a square root.)

(d) With some more work it should be possible to read off the coefficients of \(F(x)\):
\[
c_n = \frac{1}{n+1} \binom{2n}{n}.
\]