MATH 321 - HOMEWORK #7
Due Friday, March 11.

PROBLEM 1. We proved in class Peano’s theorem which states that the initial value problem
\[ x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \]
has a solution on the interval \( t \in [t_0, t_0 + \epsilon] \) for some \( \epsilon > 0 \), provided that \( f \) is continuous on \([t_0, t_0 + a] \times [x_0 - b, x_0 + b] \). The Picard-Lidelőf theorem gives a condition for uniqueness of the solution. Assume that there exists \( K \), such that
\[ |f(t, x) - f(t, y)| \leq K|x - y|, \quad t \in [t_0, t_0 + a], \quad x, y \in [x_0 - b, x_0 + b], \]
then the solution to the differential equation is unique. (The condition says that the family of functions \( \{f_t(x)\} \) is uniformly Lipschitz with the common Lipschitz constant \( K \).)

Prove the Picard-Lidelőf theorem using the following steps. Recall that we replaced the initial value problem with the problem of finding \( x(t) \), such that \( L(x)(t) = x \), where
\[ L(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds. \]
Fix \( \epsilon < \min\{a, b/M, 1/K\} \), where \( M \) is a bound for \( f \), and let \( V \subset C[t_0, t_0 + \epsilon] \) be the set of continuous functions \( x(t) \), such that \( |x(t) - x_0| \leq b \) for all \( t \in [t_0, t_0 + \epsilon] \).

(a) Prove that \( L \) maps \( V \) to \( V \).
(b) Prove that \( L \) is contracting with respect to the supremum norm:
\[ \|L(x) - L(y)\|_{\infty} < \|x - y\|_{\infty}, \]
for any \( x(t), y(t) \in V \), \( x \neq y \).
(c) Deduce that there can be at most one solution to \( L(x) = x \) in \( V \).

PROBLEM 2. Let \( f(x) \) and \( g(x) \) be continuous functions on \([a, b] \). Assume that they have equal moments:
\[ \int_{a}^{b} x^m f(x) \, dx = \int_{a}^{b} x^m g(x) \, dx, \quad m = 0, 1, 2, \ldots. \]
Prove that then \( f = g \). (Hint: A slick way to prove it is to take a sequence \( \{P_n\} \) of polynomials that uniformly approaches \( f - g \) and study the sequence \( \{(f - g)P_n\} \).)

PROBLEM 3. Prove that polynomials with rational coefficients \( \mathbb{Q}[x] \) are dense in \( C[0, 1] \) with respect to the supremum norm. In other words, for every function \( f(x) \in C[0, 1] \) and every \( \epsilon > 0 \) there exists a polynomial \( P(x) \in \mathbb{Q}[x] \), such that
\[ |f(x) - P(x)| < \epsilon, \quad \text{for all } x. \]
(This result implies that $C[0, 1]$ is a separable space, it has a countable dense subset.)

**Problem 4.** For any $f \in C[0, 1]$, define the Bernstein polynomial of degree $n$:

$$B_n(f)(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}.$$  

Bernstein’s theorem states that $B_n(f)$ uniformly approach $f$ as $n \to \infty$.

(a) Prove Bernstein’s theorem for $f(x) = x^m$, $m = 0, 1, 2, \ldots$. (Hint: Use the binomial theorem for

$$[(1 - x) + xe^t]^n.$$  

Evaluate both sides at $t = 0$ to get the case $m = 0$. Then differentiate both sides with respect to $t$ and again evaluate at $t = 0$ to get the case $m = 1$. Repeat this.)

(b) Prove Bernstein’s theorem for general $f$. (Hint: We need to show that $|B_n(f)(x) - f(x)| < \epsilon$ for all $x$ and for all $n$ large enough. Use Weierstrass theorem to find a polynomial $P(x)$ close to $f(x)$. Then use $P(x)$ and $B_n(P)(x)$ to estimate the difference.)