1. (a) The sequence \( (f_n) \) converges uniformly to \( f \) iff
\[
\sup_x |f_n(x) - f(x)| \xrightarrow{n \to \infty} 0
\]

Here
\[
\sup_{x \in [0,1]} |x^n - 0| = 1 \xrightarrow{n \to \infty} 1.
\]

(b) Let's find
\[
\sup_x |f_n(x) - f(x)| = \max_x |f_n(x)| = \max_x n x (1-x)^n.
\]

Since \( f_n(x) \) is differentiable, find critical point:
\[
0 = f_n'(x) = n (1-x)^n - n x (1-x)^{n-1} - (1-x)^n \left[ \frac{n (1-x) - n^2}{n+1} \right] = (1-x)^{n-1} \left[ 1 - (n+1)x \right]
\]
\[
\Rightarrow x = \frac{1}{n+1}
\]
\[
f_n \left( \frac{1}{n+1} \right) = n \cdot \frac{1}{n+1} \left( 1 - \frac{1}{n+1} \right)^n \xrightarrow{n \to \infty} 1 \cdot e^{-1} + C.
\]

(c) Use uniform Cauchy's criterion: \( f_n \to f \) uniformly if
\[\forall \varepsilon > 0 \exists N \text{ s.t. } \sup_x |f_n(x) - f_m(x)| < \varepsilon \text{ for } n, m \geq N.
\]

Consider the sequence of partial sums \( S_n(x) \xrightarrow{\text{e}} e^x \).
Let \( n = m + 1 \). Then
\[
\sup_x |S_n(x) - S_m(x)| = \sup_x \left| \frac{x^n}{n!} \right| = 0
\]
because \( x^n \) is not bounded. Thus, for \( \varepsilon = 1 \), there is no \( N \) such that
\[
\sup_x |S_n(x) - S_m(x)| < \varepsilon \text{ for } n, m \geq N.
\]
2. Let \( M_n \) be a bound for \( f_n \). Set \( E = 1 \). Then from uniform convergence, we know that there exists \( N \geq 0 \), such that
\[
|f_n(x) - f(x)| < 1 \quad \forall n \geq N, \forall x \in E. 
\]
Fix one \( n \), say \( n = N \). Then from triangle inequality,
\[
|f(x)| \leq |f_N(x)| + 1 \leq M_N + 1, \quad \forall x.
\]
Hence \( f(x) \) is bounded. Again by triangle inequality,
for any \( n \geq N \)
\[
|f_n(x)| \leq |f(x)| + 1 \leq M_N + 2 \quad \forall x.
\]
Now take
\[
M = \max \{ M_1, M_2, \ldots, M_N, M_N + 2 \}.
\]
This is a bound for all \( f_n \).

3. (a) Let \( P_n \) be the partition
\[
P_n = \{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{1}{2}, 1 \}.
\]
Computing the values of \( f \) at these subdivide points:
\[
f(0) = 0, \quad f(\frac{1}{n}) = \frac{1}{n}, \quad f(\frac{2}{n}) = \frac{2}{n-1}, \quad \ldots, \quad f(1) = -1
\]
The variation for this partition, we get
\[
V(P_n, f) = \left( \frac{1}{n} \right) + \left( \frac{1}{2} + \frac{1}{3} \right) + \ldots + \left( \frac{1}{n-1} + \frac{1}{n} \right) + \frac{1}{n}
\]
As \( n \to \infty \), this sum grows to \( \infty \) because the harmonic series does not converge.

(b) Take the same partitions \( P_n \) as above. Then since \( y(\frac{1}{n}) = (f(\frac{1}{n}), 0) \), we have
\[
\Delta(P_n, y) = V(P_n, f) \xrightarrow{n \to \infty} \infty.
\]
4. (a) Let $L$ be the linear function corresponding to the sequence $(1, 1, 1, \ldots)$. Then

$$L((a_n)) = \sum_{n=0}^{\infty} a_n.$$ 

Consider sequences

$$a^n = (a_0^n, a_1^n, \ldots, a_n^n, 1, 0, 0, \ldots)$$

Then $\|a^n\| = 1 \forall n$ and

$$L(a^n) = N \neq M\|a^n\| \quad \text{(for any } M)$$

There is no $M$ such that this holds for all $N$.

(b) By definition:

$$\|b\| = \sup \sum_{n=0}^{\infty} b_n a_n \quad \|a^n\| = 1$$

Since $\max|a^n| = 1$, clearly

$$\|b\| \leq \sum_{n=0}^\infty |b_n|.$$ 

Let's now consider a sequence of elements $a^n$ in $V$, such that

$$b(a^n) \to \sum_{n=0}^{\infty} |b_n|.$$ 

Let

$$a_n^N = \begin{cases} 1 & \text{if } n \leq N, b_n > 0, \\ -1 & \text{if } n \leq N, b_n < 0, \\ 0 & \text{if } n > N. \end{cases}$$

Then

$$b(a^n) = \sum_{n=0}^{N} |b_n|.$$ 

This implies that the sum is the least upper bound.

(c) An element $a \in V$ acts on an element $b \in V^*$ by

$$a(b) = b(a) = \sum_{n=0}^{\infty} a_n b_n.$$ 

We need to show that $a$ is bounded as a linear function:

$$|a(b)| \leq M \cdot \|b\| = M \cdot \sum_{n=0}^{\infty} |b_n| \quad \forall b.$$ 

Clearly, $M = \max_n \|b_n\|$ gives such a bound.

By the same argument, any sequence $C = (c_n)$, with $|c_n| \leq M^n$ for some $M$, gives a bounded linear function on $V^*$. Taking $C = (1, 1, \ldots)$ gives an element in $(V^*)^*$ that does not come from $V$. 