MATH 321 - HOMEWORK #4

Due Friday, Feb 5.

PROBLEM 1. You may assume that the following sequences and series converge on the given domain. Show that they do not converge uniformly.
   (a) \( f_n(x) = x^n \rightarrow 0 \) on \([0, 1)\)
   (b) \( f_n(x) = nx(1 - x)^n \rightarrow 0 \) on \([0, 1] \).
   (c) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \) on \(\mathbb{R} \).
   (Hint: for the last part you can use properties of exponential function. A simple proof can also be given using the uniform Cauchy’s criterion, Theorem 7.8.)

PROBLEM 2. Recall that a function \( f(x) \) defined on \( E \) is bounded if there exists an \( M \geq 0 \), such that \( |f(x)| \leq M \) for all \( x \in E \). A sequence of functions \( \{f_n\} \) on \( E \) is uniformly bounded if there exists one bound \( M \) that works for all \( f_n \). Prove that if the functions \( f_n \) are bounded and the sequence \( f_n \) converges to some function \( f \) uniformly, then the sequence is uniformly bounded.

PROBLEM 3. This problem is about variation, and more generally, the length of curves.
   (a) Prove that \( f : [0, 1] \rightarrow \mathbb{R} \),

   \[
   f(x) = \begin{cases} 
   x \cos \frac{\pi}{x} & \text{if } 0 < x \leq 1, \\
   0 & \text{if } x = 0,
   \end{cases}
   \]

   does not have bounded variation. (Hint: consider partitions with subdivision points \( 1/n \).)
   (b) Consider the parametrized curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \),

   \[
   \gamma(x) = \begin{cases} 
   (x \cos \frac{\pi}{x}, x \sin \frac{\pi}{x}) & \text{if } 0 < x \leq 1, \\
   (0, 0) & \text{if } x = 0.
   \end{cases}
   \]

   Show that the curve is not rectifiable, \( \Lambda \gamma = \infty \).

PROBLEM 4. The Riesz representation theorem we discussed in class describes all bounded functionals on the space \( C[a, b] \). This problem studies the simplest infinite dimensional vector space \( V \), its dual \( V^* \) and the bounded dual \( V^b \).
Let \( V \) be the vector space of sequences of real numbers \( (a_n) \), such that only a finite number of \( a_n \) are nonzero,

\[
V = \{ (a_0, a_1, a_2, \ldots) | a_n = 0 \text{ for } n \text{ large enough} \}.
\]
The vector space $V$ has a countable basis
\[ e_0 = (1,0,0,...), \quad e_1 = (0,1,0,0,...), \quad e_2 = (0,0,1,0,...), \quad .... \]
To define a linear function $L : V \to \mathbb{R}$, we need to specify the values of $L$ on the basis:
\[ L(e_0) = b_0, \quad L(e_1) = b_1, \quad L(e_2) = b_2, \quad .... \]
This way we can identify the dual space $V^*$ with the space of all sequences $(b_n)$, with no restriction on vanishing. A linear function $(b_n)$ acts on $V$ by dot product.
Let $V$ have the norm
\[ \|(a_n)\| = \max_n |a_n|. \]
Recall that a linear function $L \in V^*$ is bounded if there exists an $M \geq 0$, such that for every $v = (a_n) \in V$,
\[ |L(v)| \leq M\|v\|. \]
The smallest such bound is defined to be the norm of $L$:
\[ \|L\| = \sup_{v \neq 0} \frac{|L(v)|}{\|v\|} = \sup_{\|v\|=1} |L(v)|. \]
(a) Show that the linear function
\[ (1,1,...) \in V^* \]
is not bounded.
(b) For any $b = (b_n) \in V^*$, show that
\[ \|b\| = \sum_n |b_n|. \]
In particular, the subspace of bounded linear functions $V^b \subset V^*$ is the set of sequences $(b_n)$ such that $\sum_n |b_n| < \infty$.
(c) Show that every element of $V$ defines (by dot product) a bounded linear function $V^b \to \mathbb{R}$, giving a linear map:
\[ V \to (V^b)^b, \]
but this map is not surjective. For example, the sequence $(1,1,...)$ defines element of $(V^b)^b$ that does not come from $V$. 