1. Let $P_k$ be the partition into $2k-1$ pieces of equal length. Then $1$ lies in the middle of $[x_{k-1}, x_k]$. Choose $x_k > 1$, $x_k < 1$ and the other $x_i$, $s_i$ arbitrary. Then
\[
\lim_{k \to \infty} R S(P_k, \{x_i\}, f, x) = f(x_k) \cdot \Delta x_k = 1 \cdot 1 \to 1;
\]
\[
\lim_{k \to \infty} R S(P_k, \{s_i\}, f, x) = f(s_k) \cdot \Delta x_k = 0 \cdot 1 \to 0.
\]

2. Let $P$ have our partition $[0, \delta]$. We know that $f$ is $2k_i$ on $[\delta, 1]$ because $f$ has a finite number of discontinuities and $\delta$ is continuous at these points. Hence we can find a partition of $[\delta, 1]$ so that the upper and lower sums on $[\delta, 1]$ are less than $\epsilon^2$. On $[0, \delta]$ since $f$ is bounded, $|f| \leq B$, the difference between the upper and lower sum is at most
\[
\Delta x \cdot (M_i - m_i) = (\delta(\delta) - \delta(0)) \cdot 2B.
\]
Since $\delta$ is continuous at $0$, we can choose $\delta$ such that $\delta(\delta) - \delta(0) < \epsilon$. Hence the difference between the lower and upper sum $\delta$ on $[0, 1]$ is less than
\[
\epsilon \cdot 2B + \epsilon = \epsilon (2B + 1),
\]
which can be made arbitrarily small by choosing $\epsilon$ small.
3. (a) We have (since \( \alpha \) and \( f \) are nondecreasing)

\[
U(P, \alpha, \phi) = \sum_{i=1}^{n} \phi(x_i) \cdot [\alpha(x_i) - \alpha(x_{i-1})] = \sum_{i=1}^{n} \phi(x_i) \cdot \alpha(x_i) - \sum_{i=1}^{n} \phi(x_i) \cdot \alpha(x_i)
\]

\[
L(P, \alpha, \phi) = \sum_{i=1}^{n} \alpha(x_{i-1}) \cdot [\phi(x_i) - \phi(x_{i-1})]
\]

\[
= \sum_{i=1}^{n} \phi(x_i) \cdot \alpha(x_{i-1}) - \sum_{i=0}^{n-1} \phi(x_i) \cdot \alpha(x_i)
\]

The sum of these is

\[
\phi(x_n) \cdot \alpha(x_n) - \phi(x_0) \cdot \alpha(x_0).
\]

(b) Subtract the equations

\[
U(P, \alpha, \phi) + L(P, \alpha, \phi) = X
\]

\[
L(P, \alpha, \phi) + U(P, \alpha, \phi) = X
\]

to get

\[
U(P, \alpha, \phi) - L(P, \alpha, \phi) = U(P, \phi, \alpha) - L(P, \alpha, \phi).
\]

If \( f \in \mathcal{F} \), then \( \exists \varepsilon \neq P \) such that the left hand side is less than \( \varepsilon \). Then also the right hand side is \( \varepsilon \) and hence \( \alpha \in \mathcal{F}(f) \).

Now

\[
L(P, \alpha, \phi) = \int_{a}^{b} \phi(b) \cdot d\lambda - \int_{a}^{b} \phi(a) \cdot d\lambda - U(P, \phi, \alpha)
\]

Here

\[
\sup_{P} L(P, \alpha, \phi) = \int_{a}^{b} \phi(b) \cdot d\lambda - \int_{a}^{b} \phi(a) \cdot d\lambda - \inf_{P} U(P, \phi, \alpha)
\]

\[
\int_{a}^{b} \phi(b) \, dx
\]

\[
\int_{a}^{b} \phi(a) \, dx
\]
4. Let $I_k = [a+1, a+k+1]$. We know that
\[ \int_a^{a+k} f \, dx \]
exists, hence it equals to \((\text{Thm 6.12 (c)})\)
\[ \int_a^{a+1} f \, dx + \int_{a+1}^{a+2} f \, dx + \ldots + \int_{a+k}^{a+k+1} f \, dx. \]
Since
\[ \int_a^{a+k} f \, dx \quad \text{as} \quad k \to \infty \]
we get
\[ \int_a^\infty f \, dx = \int_a^{a+1} f \, dx + \int_{a+1}^{a+2} f \, dx + \ldots \]
Choose a partition $P_k$ of each $[a+k, a+k+1]$ and combine these
to a partition $P$ of $[a, \infty)$. Since $f \in \mathcal{R}(\xi)$ on $[a, \infty)$,
we can choose $P_k$ such that
\[ U(P_k, f, x) + L(P_k, f, x) < \frac{\epsilon}{2^k}. \]
Then also
\[ U(P_k, f, x) - \int_{a+k}^{a+1} f \, dx < \frac{\epsilon}{2^k}, \]
\[ \int_{a+k}^{a+1} f \, dx - L(P_k, f, x) < \frac{\epsilon}{2^k}. \]
Now
\[ U(P_k, f, x) = \sum_{k=a}^{a+k} U(P_k, f, x) < \sum_{k=a}^{a+k} \int_{a+k}^{a+1} f \, dx < \frac{\epsilon}{2^k} = \int_a^\infty f \, dx + 2\epsilon. \]
Similarly,
\[ L(P_k, f, x) > \int_a^\infty f \, dx - 2\epsilon. \]
In particular, both the upper sum and the lower sum are finite.

We also have
\[ U(P_k, f, x) - L(P_k, f, x) < 4\epsilon, \]
which can be made arbitrarily small by taking $\epsilon$ small.

Let $\epsilon$ approach zero, then
\[
\int_a^\infty f \, dx = \inf \int_a^\infty f \, dx \leq \int_a^\infty f \, dx \leq \sup \int_a^\infty f \, dx = \frac{\int_a^\infty f \, dx}{2^k},
\]
hence all integrals are equal.