MATH 321 - HOMEWORK #1

Due Friday, Jan 15.

PROBLEM 1. Recall that we defined the Riemann sum
\[ RS(P, \{ t_i \}, f, \alpha) = \sum_{i=1}^n f(t_i) \Delta \alpha_i. \]
This sum depends on the partition \( P \), the tagging \( t_i \in [x_{i-1}, x_i] \), the functions \( f \) and \( \alpha \). We also defined the mesh of a partition \( P \) to be
\[ \| P \| = \max_i (x_i - x_{i-1}). \]
Let \( f, \alpha : [0,2] \to \mathbb{R} \) be as in Homework 1, Problem 2:
\[ f(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{otherwise}, \end{cases} \quad \alpha(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{otherwise}. \end{cases} \]
Find a sequence of partitions \( P_1, P_2, P_3, \ldots \), with \( \lim_{k \to \infty} \| P_k \| = 0 \), and two different taggings for each partition \( \{ t_i \} \) and \( \{ s_i \} \), such that
\[ \lim_{k \to \infty} RS(P_k, \{ t_i \}, f, \alpha) = 1, \quad \lim_{k \to \infty} RS(P_k, \{ s_i \}, f, \alpha) = 0. \]
This problem shows that, even though \( f \in R(\alpha) \), we cannot compute the integral by arbitrary Riemann sums.

PROBLEM 2. Let \( f, \alpha : [0,1] \to \mathbb{R} \), with \( f \) bounded and \( \alpha \) nondecreasing. Assume that \( f \) is continuous except at the points in \( S = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \} \) and that \( \alpha \) is continuous at all points of \( S \). Prove that then \( f \in R(\alpha) \).
(Hint: Divide the domain \([0,1]\) into two intervals \([0, \delta] \cup [\delta,1]\) for some small \( \delta \) and make the difference between the upper and lower sums small on each interval separately.)

PROBLEM 3. Let \( f, \alpha : [a,b] \to \mathbb{R} \) be nondecreasing functions. The goal of this problem is to prove the integration by parts formula for Riemann-Stieltjes integrals:
\[ \int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a). \quad (\ast) \]
This formula should be read as: if one integral exists, then so does the other and then the equality holds.
1) Prove that for any partition \( P \),
\[ U(P, f, \alpha) + L(P, \alpha, f) = f(b)\alpha(b) - f(a)\alpha(a). \]
(Then also, interchanging \( f \) and \( \alpha \),
\[ L(P, f, \alpha) + U(P, \alpha, f) = f(b)\alpha(b) - f(a)\alpha(a). \]

(2) Show that if \( f \in \mathcal{R}(\alpha) \), then \( \alpha \in \mathcal{R}(f) \) and the equality \((*)\) holds.

**Problem 4.** Let \( f, \alpha : [a, \infty) \to \mathbb{R} \) and assume that \( f \in \mathcal{R}(\alpha) \) on any finite interval \([a, b], a \leq b \). Define the improper integral
\[
\int_{a}^{\infty} f \, d\alpha = \lim_{b \to \infty} \int_{a}^{b} f \, d\alpha
\]
if this limit exists. Assume that the improper integral exists and prove that it can be computed using upper and lower sums of partitions of \([a, \infty)\).

More precisely, define a partition \( P \) of \([a, \infty)\) to be a sequence \( \{a = x_0 < x_1 < x_2 < \ldots\} \), where \( \lim_{i \to \infty} x_i = \infty \). For each partition define the upper and lower sums the same way as before. These sums may be infinite. Define the upper and lower integrals as before, using infimum and supremum over all partitions of \([a, \infty)\). Prove that if \( f \) is integrable on \([a, \infty)\) according to the limit definition above, then
\[
\int_{a}^{\infty} f \, d\alpha = \int_{a}^{\infty} f \, d\alpha = \int_{a}^{\infty} f \, d\alpha.
\]

(Hint: Divide \([a, \infty)\) into a sequence of finite intervals, for example \( I_k = [a + k, a + k + 1] \), \( k = 0, 1, \ldots \). It should follow from what we prove in class that the improper integral is the sum of integrals over these finite intervals. Now construct a partition that respects the subdivision into \( I_k \), where the difference between the upper and lower sum is small on each \( I_k \), and moreover, these differences sum up to \( \varepsilon \). The same method can be used to show that the upper and lower sums are finite for some \( P \).)