Math 321, Exam #2, Solutions.

1. (a) \( \{f_n\} \) is not equicontinuous:
\[
|f_n(x) - f_n(y)| = 1, \quad |x - \frac{1}{n}| \to 0.
\]

(b) \( \{g_n\} \) is equicontinuous. \( g_n \xrightarrow{n \to \infty} 0 \) uniformly, hence we can apply the converse of Arzela-Ascoli theorem.

2. We need to show that \( \{f_n\} \) is equi-continuous and uniformly bounded.

Uniformly bounded:
\[
|f_n(x)| \leq \int |\sin(t-x)| \cdot M \, dx \leq M, \quad \text{where} \quad |\phi(x)| \leq M \, \forall x.
\]

Equi-continuous:
\[
|f_n(x) - f_n(y)| \leq \int |\sin(t-x) - \sin(t-y)| \cdot |\phi(t)| \, dx < \varepsilon
\]
\[
\text{if} \quad |x-y| < \delta
\]

How we used:
\[
|\sin(t-x) - \sin(t-y)| = |\sin'(t-c) \cdot (x-y)| = |\cos(t-c)| \cdot |x-y| \leq |x-y|
\]

Now we can apply Arzela-Ascoli theorem.
3. The assumption implies that if \( p(x) \) is a polynomial, then
\[
\int f(x) \cdot p(x^2) \, dx = 0.
\]
We need to show that \( f(x) \) is a uniform limit of polynomials \( p_n(x^2) \xrightarrow{n \to \infty} f(x) \). Thus proceed as in the homework.

The function \( g(x) = x^2 \) is strictly increasing on \([-1,1]\), hence by homework, \( R[g] = R[\mathbb{R}^2] \) is dense in \( C([-1,1]) \).

4. Consider \( g(x) = f(x) - \frac{\varepsilon}{2} \). By Weierstrass theorem, there exists a polynomial \( p(x) \), such that
\[
|g(x) - p(x)| < \frac{\varepsilon}{2} \quad \forall x.
\]

Hence
\[
|f(x) - p(x)| \leq |f(x) - g(x)| + |g(x) - p(x)| < \varepsilon.
\]

Clearly \( f(x) > p(x) \quad \forall x \).

The polynomial \( p(x) \) may take negative values. To avoid this, replace \( \varepsilon \) by
\[
\min \{ \varepsilon, \min_{x \in [-1,1]} f(x) \}
\]
and start from the beginning. Now \( g(x) > \frac{\varepsilon}{2} \quad \forall x \), hence \( p(x) > 0 \quad \forall x \).