1. Calculate the Fourier coefficients \((c_n, a_n, b_n)\) for the triangle function

\[ f(t) = \begin{cases} 
2t & \text{if } 0 \leq t \leq 1/2 \\
2 - 2t & \text{if } 1/2 \leq t \leq 1 
\end{cases} \]

and show that the Fourier series decomposition of \(f(t)\) may be written

\[ f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi nt) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi^2 (2n+1)^2} \cos(2\pi (2n+1)t) \]

What does Parseval’s formula say in this case?

We compute

\[ c_0 = \int_0^{1/2} 2tdt + \int_{1/2}^1 (2-2t)dt = 1/2. \]

For \(n \neq 0\) we can use integration by parts

\[
\int e^{-2\pi int}tdt = \frac{te^{-2\pi int}}{-2\pi in} - \int \frac{e^{-2\pi int}}{-2\pi in} dt \\
= \frac{te^{-2\pi int}}{-2\pi in} + \frac{e^{-2\pi int}}{(2\pi n)^2}
\]

and the fact that \(e^{-i\pi} = -1\) to conclude

\[
c_n = \int_0^{1/2} e^{-2\pi int} 2tdt + \int_{1/2}^1 e^{-2\pi int}(2-2t)dt \\
= \frac{((-1)^n - 1)}{\pi^2 n^2} \\
= \begin{cases} 
0 & \text{if } n \text{ is even, } n \neq 0 \\
-2/(\pi^2 n^2) & \text{if } n \text{ is odd}
\end{cases}
\]

This leads to

\[ a_0 = 2c_0 = 1 \]

\[ a_n = \begin{cases} 
0 & \text{if } n \text{ is even, } n \neq 0 \\
-\frac{4}{\pi^2 n^2} & \text{if } n \text{ is odd}
\end{cases} \]

\[ b_n = 0 \]

Thus the real form of the Fourier series for \(f(t)\) is

\[ f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi nt) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi^2 (2n+1)^2} \cos(2\pi (2n+1)t) \]
Since

$$\int_0^1 f^2(t)dt = \int_0^{(1/2)} (2t)^2dt + \int_{(1/2)}^1 (2-2t)^2dt = \frac{1}{3}$$

Parseval’s formula says:

$$\frac{1}{4} + \sum_{n=-\infty}^{\infty} \frac{4}{\pi^4 n^4} = \frac{1}{3}$$

or

$$\frac{8}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{12}$$

or

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

2. Modify the file ftdeom01.m so that it plots the partial sums of the Fourier series in the previous question. Hand in the code and a plot of the partial sums with 1, 2, 5 and 10 non-zero terms.

Here is the code for the function

```matlab
function ftdemo3(N)
    % Fourier series for f(x) = 2x for 0 < x < 0.5
    % = 2-2x for 0.5 < x < 1
    X=linspace(0,1,1000);
    F=0.5*ones(1,1000);
    for n=[0:N]
        F = F - 4*cos(2*pi*(2*n+1)*X)/(pi^2*(2*n+1)^2);
    end
    plot(X,F)
    axis([-0.2,1.2,-0.1,1.1])
end
```

and the results of running `ftdemo3(N)` for $N = 0, 1, 4, 9$ (the number of nonzero terms is $N + 1$)
3. Calculate the Fourier coefficients \((c_n, a_n, b_n)\) for the half sine wave

\[ f(t) = \sin(\pi t) \quad \text{for} \quad 0 \leq t \leq 1 \]

and show that the Fourier series for \(f(t)\) can be written

\[ f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos(2\pi nt) \]

For \(n = 0\) we find

\[ c_0 = \int_0^1 \sin(\pi t)dt = \left[ -\frac{1}{\pi} \cos(\pi t) \right]_0^1 = \frac{2}{\pi}. \]

For \(n \neq 0\) it is easier to expand \(\sin(\pi t) = \frac{1}{2i}(e^{\pi it} - e^{-\pi it})\). We find

\[ c_n = \int_0^1 e^{-2\pi int/\pi} \frac{1}{2i}(e^{\pi it} - e^{-\pi it})dt \]

\[ = \frac{1}{2i} \int_0^1 \left( e^{\pi i(1-2n)t} - e^{-\pi i(1+2n)t} \right)dt \]

\[ = \frac{1}{2i} \left[ \frac{1}{\pi i(1-2n)} e^{\pi i(1-2n)t} + \frac{1}{\pi i(1+2n)} e^{-\pi i(1+2n)t} \right]_0^1 \]

\[ = \frac{1}{2i} \left( \frac{1}{\pi i(1-2n)}(e^{\pi i(1-2n)} - 1) + \frac{1}{\pi i(1+2n)}(e^{-\pi i(1+2n)} - 1) \right) \]

Now \(e^{2\pi i} = 1\) and \(e^{\pi i} = -1\), so

\[ c_n = \frac{1}{2i} \left( \frac{1}{\pi i(1-2n)}(-2) + \frac{1}{\pi i(1+2n)}(-2) \right) \]

\[ = \frac{1}{\pi(1-2n)} + \frac{1}{\pi(1+2n)} = \frac{2}{\pi(1-4n^2)}. \]

Using the relations between \(c_n\) and \(a_n, b_n\) we find:

\[ a_0 = 2c_0 = \frac{4}{\pi}, \quad a_n = c_n + c_{-n} = \frac{4}{\pi(1-4n^2)}, \quad b_n = i(c_n - c_{-n}) = 0 \]

and the series can be written

\[ f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos(2\pi nt). \]
4. Calculate the Fourier coefficients \((c_n's, a_n's \text{ and } b_n's)\) for the function

\[ f(t) = t^2 - 1 \text{ for } -1 \leq t \leq 1 \]

and show that the Fourier series for \(f(t)\) can be written

\[ f(t) = -\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t). \]

Here the time period \(T\) is 2.

For \(n = 0\) we have

\[ c_0 = \frac{1}{2} \int_{-1}^{1} (t^2 - 1) dt = \frac{2}{3}. \]

For \(n \neq 0\) we have

\[
\begin{aligned}
c_n &= \frac{1}{2} \int_{-1}^{1} e^{-\pi i n t/2} (t^2 - 1) dt \\
&= \frac{1}{2} \left[ \left( \frac{1}{-\pi in} e^{-\pi int} (t^2 - 1) \right)_{-1}^{1} - \int_{-1}^{1} \frac{1}{-\pi in} e^{-\pi int} 2tdt \right] \text{ (integration by parts)} \\
&= -\frac{1}{2} \left[ \left( \frac{1}{(-\pi in)^2} e^{-\pi int} 2t \right)_{-1}^{1} + \frac{1}{2} \int_{-1}^{1} \frac{1}{(-\pi in)^2} e^{-\pi int} 2dt \right] \text{ (integration by parts again)} \\
&= \frac{1}{\pi^2 n^2} (e^{-\pi in} + e^{\pi in}) + \frac{1}{2} \left[ \frac{1}{(-\pi in)^3} e^{-\pi int} \right]_{-1}^{1}
\end{aligned}
\]

But \(e^{\pi in} = e^{-\pi in} = (-1)^n\) so

\[
c_n = \frac{1}{\pi^2 n^2} ((-1)^n + (-1)^n) + \frac{1}{(-\pi in)^3} ((-1)^n - (-1)^n) = \frac{2(-1)^n}{\pi^2 n^2}
\]

Using the relations between \(c_n\) and \(a_n, b_n\) we find:

\[
a_0 = 2c_0 = -\frac{4}{3}, \quad a_n = c_n + c_{-n} = \frac{4(-1)^n}{\pi^2 n^2}, \quad b_n = i(c_n - c_{-n}) = 0,
\]

and the series can be written

\[ f(t) = -\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t). \]

5. Show that the Fourier series of \(f(t) = e^t\) on the interval \(-\pi \leq t \leq \pi\) is

\[ f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1-i n} (e^{1-in}\pi - e^{-(1-in)}\pi) e^{int}. \]

Deduce that

\[ \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} (\pi \coth \pi - 1). \]

Here the period \(T\) is \(2\pi\).
For any $n$ we find
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2\pi i t/2\pi} e^{t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{t(1-in)} dt
\]
\[
= \frac{1}{2\pi} \left[ \frac{1}{1-in} e^{t(1-in)} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left( \frac{1}{1-in} e^{\pi(1-in)} - \frac{1}{1-in} e^{-\pi(1-in)} \right)
\]
\[
= \frac{1}{2\pi} \frac{1}{1-in} \left( e^{\pi(1-in)} - e^{-\pi(1-in)} \right)
\]
and so the Fourier decomposition can be written
\[
f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1-in} \left( e^{(1-in)\pi} - e^{-(1-in)\pi} \right) e^{int}.
\]

Now we can use Parseval’s formula to find the required expression as follows
\[
\frac{1}{2\pi} \langle e^t, e^t \rangle = \sum_{n=-\infty}^{\infty} |c_n|^2
\]
The inner product on the left hand side is
\[
\langle e^t, e^t \rangle = \int_{-\pi}^{\pi} e^{2t} dt = \frac{1}{2} \left( e^{2\pi} - e^{-2\pi} \right) = \frac{1}{2} (e^{\pi} + e^{-\pi})(e^{\pi} - e^{-\pi}).
\]
The terms in the sum on the right hand side are
\[
|c_n|^2 = \frac{1}{4\pi^2} \frac{1}{1+n^2} (e^{\pi} - e^{-\pi})^2
\]
where we have used the fact that $e^{\pm in\pi} = (-1)^n$.

Putting these together we have
\[
\frac{1}{2\pi} \frac{1}{2} (e^{\pi} + e^{-\pi})(e^{\pi} - e^{-\pi}) = \frac{1}{4\pi^2} (e^{\pi} - e^{-\pi})^2 \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2}.
\]
This becomes
\[
(e^{\pi} + e^{-\pi}) = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) \left( 2 \sum_{n=1}^{\infty} \frac{1}{1+n^2} + 1 \right)
\]
Using coth $\pi = (e^{\pi} + e^{-\pi})/(e^{\pi} - e^{-\pi})$ we have the final result
\[
\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} (\pi \coth \pi - 1).
\]