MATH 307 - 202
MIDTERM EXAM

Name:
Student ID:

Exam rules:

• No calculators, open books, or notes are allowed.
• Use back sides of pages if needed.
• This exam has 7 problems.
• A sheet with Matlab formulas is attached at the end.
Problem 1. (6 pts.) Let \( A \) be a \( 3 \times 4 \) matrix such that

\[
\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.
\]

(a) Find the dimension and a basis for \( N(A) \).
(b) Find the dimension and a basis for \( N(A^T) \).
(c) Find the dimension and a basis for \( R(A) \).

\[
(a) \quad \begin{cases} x_1 - 2x_2 + 3x_4 = 0 \\ x_3 - x_4 = 0 \end{cases}
\]

\[
\text{dim } N(A) = 2
\]

\[
(b) \quad \begin{align*}
&x_1 = t \\ &x_2 = 2t \\ &x_5 = \text{free}
\end{align*}
\]

\[
\text{basis for } N(A^T)
\]

\[
\text{dim } N(A^T) = 1.
\]

\[
(c) \quad R(A) = N(A^T)^\perp = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + 2x_2 + x_3 = 0 \right\}
\]

\[
\begin{align*}
&x_1 = -2t - s \\ &x_2 = t \\ &x_3 = s
\end{align*}
\]

\[
\text{basis for } R(A)
\]

\[
\text{dim } R(A) = 2.
\]

Another way:

\[
R(A) = \text{Row space } (A^T) = \text{Row space } (\text{rref } A^T)
\]

basis: \#0 rows of \( \text{rref}(A^T) \):

\[
\begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \end{bmatrix}
\]
PROBLEM 2. (4 pts.) Which of the following sets are subspaces? No proof is needed, but you may want to sketch a proof to convince yourself.

(a) All vectors \([x, y, z]^T\) in \(\mathbb{R}^3\) such that \(x + y + z = 1\).

\[\text{Not subspace}\]

(b) All \(2 \times 2\) matrices \(A\) such that \(AB = B^T A^T\). Here \(B\) is some fixed \(2 \times 2\) matrix.

\[\text{Subspace}\]

(c) All continuous functions \(f(x)\) defined on the interval \([0, 10]\) such that on each interval \([i - 1, i]\) for \(i = 1, \ldots, 10\), \(f(x)\) is equal to a degree 3 polynomial. (This is a subset of all continuous functions on \([0, 10]\).)

\[\text{Subspace}\]

(d) All solutions to the differential equation \(f''(x) = x^2 f(x)\). (As a subset of all functions on \(\mathbb{R}\).)

\[\text{Subspace}\]
Problem 3. (4 pts.) Consider the system of linear equations \( A\vec{x} = \vec{b} \) for some invertible matrix \( A \) and vector \( \vec{b} \). Assume that \( A, \vec{b} \) and the solution vector \( \vec{x} \) satisfy:

\[
\|\vec{b}\| = 2, \quad \|\vec{x}\| = 7, \quad \|A\| = 3, \quad \|A^{-1}\| = 5.
\]

When we change \( \vec{b} \) by \( \Delta \vec{b} \), then \( \vec{x} \) changes by \( \Delta \vec{x} \).

(a) How large can the norm of \( \Delta \vec{b} \) be so that the norm of \( \Delta \vec{x} \) is no bigger than \( \varepsilon \)?

(b) How large can the norm of \( \Delta \vec{b} \) be so that the relative error of \( \vec{x} \) is no bigger than \( \varepsilon \)?

\[(a) \quad \Delta \vec{x} = A^{-1} \Delta \vec{b} \quad \text{want}
\]
\[
\|\Delta \vec{x}\| \leq \|A^{-1}\| \cdot \|\Delta \vec{b}\| \leq \varepsilon
\]
\[
\|\Delta \vec{b}\| \leq \frac{\varepsilon}{\|A^{-1}\|} = \frac{\varepsilon}{5}
\]

\[(b) \quad \frac{\|\Delta \vec{x}\|}{\|\vec{x}\|} \leq \text{cond} (A) \cdot \frac{\|\Delta \vec{b}\|}{\|\vec{b}\|} \leq \varepsilon
\]
\[
\|\Delta \vec{b}\| \leq \frac{\|\vec{b}\| \cdot \varepsilon}{\|A\| \cdot \|A^{-1}\|} = \varepsilon \cdot \frac{2}{15}
\]
**Problem 4.** (4 pts.) Let $A$ and $B$ be $2 \times 2$ matrices with columns switched:

$$A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}, \quad B = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$ 

Are the following statements true? Explain briefly.

(a) The Hilbert-Schmidt norms of $A$ and $B$ are equal.
(b) The operator norms of $A$ and $B$ are equal.

\begin{align*}
\text{(a) True:} \\
\text{H.S. norm:} \\
&\|\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\|_{HS} = \sqrt{v_1^2 + v_2^2 + w_1^2 + w_2^2} \\
&\|\begin{bmatrix} w_1 & v_1 \\ w_2 & v_2 \end{bmatrix}\|_{HS} = \sqrt{w_1^2 + w_2^2 + v_1^2 + v_2^2}.
\end{align*}

\begin{align*}
\text{(b) True.} \\
\|A\| = \max_{\|\vec{u}\|=1} \|A\vec{u}\| = \max_{\|\vec{u}\|=1} \|m_1 \vec{v} + m_2 \vec{w}\| &\overset{\text{same exposic}}{\leq} \max_{\|\vec{u}\|=1} \|m_1 \vec{v} + m_2 \vec{w}\| &\overset{\text{switch } m_1, m_2}{=} \max_{\|\vec{u}\|=1} \|m_1 \vec{w} + m_2 \vec{v}\| \\
\|B\| = &\max_{\|\vec{u}\|=1} \|m_1 \vec{w} + m_2 \vec{v}\| \\
\text{Another way:} \\
B = A \cdot C \\
C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &\text{reflection across line } x = y. \\
\|B\| = \|A \cdot C\| = \max_{\|\vec{u}\|=1} \|AC\vec{u}\| = \max_{\|\vec{v}\|=1} \|A\vec{v}\| = \|A\| \\
C \text{ does not change norm,} \\
\|C\vec{u}\| = \|\vec{u}\|.
\end{align*}
Problem 5. (8 pts.) Consider the oriented graph with $n$ vertices and $n$ edges shown below. Assume $n > 3$.

(a) Find the incidence matrix $D$ of the graph. Show at least the first three and the last two rows.

(b) (i) Find the rank of $D$.
(ii) Find the rank of the incidence matrix if we remove the edge labeled $n$ from the graph.

(c) Write down Matlab commands to enter the matrix $D$. Assume that $n$ is defined already.

\[
D = \begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & -1 \\
1 & 1 & 1 \\
\vdots \\
1 & 1 & 1 \\
-1 & 1 & -1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

(b) (i) $\dim N(D) = 1$ = # connected comp.
$\text{Rank}(D) = n-1$

(ii) Remove last row from $D$, result in echelon form, $n-1$ pivots
$\text{Rank}(\text{new } D) = n-1$.

(c) $A = -\text{eye}(n)$;
$B = \text{diag(ones(1,n-1), 1)}$;
$D = A + B$;
$D(n, 1) = 1$. 

\[
\text{Diagram of oriented graph}
\]
PROBLEM 6. (8 pts.) Consider the differential equation for the unknown function \( f(x) \) defined on the interval \([0, 2]\):

\[
f''(x) + f(x) = 2x, \quad f(0) = 0.
\]

To solve this equation numerically, we choose equally spaced subdivision points on \([0, 2]\), \(0 = x_1, x_2, \ldots, x_N = 2\), and let \( f_i = f(x_i) \).

(a) Find a linear equation in \( f_i \) that describes the differential equation at the point \( x_i \) for \( i = 1, \ldots, N - 1 \). Use \((f(x_{i+1}) - f(x_i))/\Delta x\) to approximate the first derivative of \( f \) at \( x_i \).

(b) Find the matrix form of the system of equations that we need to solve. Use the initial condition as the first row in the equation. Show at least the first three and the last two rows.

(c) Write down Matlab commands to enter the matrix \( A \) and the right hand side \( b \) of the system. Assume that \( N > 0 \) is defined. Start with finding \( X = [x_1, \ldots, x_N] \), \( L = \Delta x \), and so on.

\[
\begin{align*}
\frac{f_{i+1} - f_i}{\Delta x} + f_i &= 2x_i, \\
\left(\Delta x - 1\right) f_i + f_{i+1} &= 2x_i \Delta x.
\end{align*}
\]

\[
\begin{bmatrix}
1 \\
\Delta x^{-1} & 1 \\
\Delta x^{-1} & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_{n-1} \\
f_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
2x_1 \Delta x \\
2x_2 \Delta x \\
\vdots \\
2x_{n-1} \Delta x \\
2x_n \Delta x
\end{bmatrix}
\]

\[
X = \text{linspace}(0, 2, N);
\]

\[
L = X(2) - X(1);
\]

\[
A = \text{eye}(n) + (L-1) \times \text{diag}([1, \ldots, 1, -1]);
\]

\[
b = 2 \times L \times [0 \times X(1, 2:\text{n-1})]';
\]
PROBLEM 7. (8 pts.) Let \( f(x) \) be a function defined on the interval \([-1, 1]\),
\[
f(x) = \begin{cases} 
    a_1 + b_1 e^x + c_1 e^{2x} & \text{for } x \in [-1, 0], \\
    a_2 + b_2 e^x + c_2 e^{2x} & \text{for } x \in [0, 1]. 
\end{cases}
\]
We also require that
\[
f(-1) = 3, \quad f(0) = 7, \quad f(1) = 4,
\]
and that \( f(x) \) is two times differentiable at all points \( x \) in \((-1, 1)\).

We want to determine the unknown coefficients \( a_1, b_1, c_1, a_2, b_2, c_2 \).

(a) Find equations stating that \( f(x) \) takes the correct value at points \(-1, 0, 1\).

(b) Find equations stating that \( f(x) \) has first and second derivatives.

(c) Write the resulting system of equations in matrix form.

(a) \[
\begin{align*}
    a_1 + b_1 e^-1 + c_1 e^{-2} &= 3 \\
    a_1 + b_1 + c_1 &= 7 \\
    a_2 + b_2 + c_2 &= 7 \\
    a_2 + b_2 e^1 + c_2 e^2 &= 4
\end{align*}
\]

(b) \[
\begin{align*}
    b_1 + 2c_1 &= b_2 + 2c_2 \quad \Leftrightarrow \quad b_1 + 2c_1 + b_2 - 2c_2 = 0 \\
    b_1 + 4c_1 &= b_2 + 4c_2 \quad \Leftrightarrow \quad b_1 + 4c_1 - b_2 - 4c_2 = 0
\end{align*}
\]

(c) \[
\begin{bmatrix}
1 & e^-1 & e^{-2} \\
1 & 1 & 1 \\
1 & e & e^2 \\
1 & 2 & -1 & -2 \\
1 & 4 & -1 & -4
\end{bmatrix} \begin{bmatrix}
a_1 \\
b_1 \\
c_1 \\
a_2 \\
b_2 \\
c_2
\end{bmatrix} = \begin{bmatrix}
3 \\
7 \\
7 \\
4 \\
0 \\
0
\end{bmatrix}
\]