Part A - Short Answer Questions, 1 mark each

For Questions A1-A2 below, let

\[ a = [1, 1, 5] \]
\[ b = [2, -2, 1] \]

A1: Calculate \( \|a\| \).

\[ \sqrt{27} \]

A2: Calculate \( a \cdot b \).

\[ 5 \]

A3: Find a normal vector to the plane given by parametric equation

\[ x = [1 + 2s + t, s - t, 2 - s + t], \]

\[ = (1, 0, 2) + s (2, 1, -1) + t (1, -1, 1) \]

\( \text{normal} = (2, 1, -1) \times (1, -1, 1) \)

\[ = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (0, -3, -3). \]

A4: Find all values of \( w \) such that the vectors \( a = [1, -2, 3] \) and \( b = [1, w, -2] \) are orthogonal.

\[ a \cdot b = 1 - 2w - 6 = 0 \implies -2w = 5, \ w = -5/2. \]

A5: Consider the following lines of MATLAB code:

\[ A = zeros(3, 4); \]
\[ A(2, 1) = 5; \]
\[ A(1:3, 4) = [3 2 1]; \]

What is the result in \( A \)?

\[
\begin{pmatrix}
0 & 0 & 0 & 3 \\
5 & 0 & 0 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
A6: Consider a linear system of six equations in eight unknowns. Circle below all possible forms of the solution set to the system.

(a) The system might have no solutions
(b) The system might have exactly one solution
(c) The system might have exactly two solutions
(d) The system might have an infinite number of solutions
(e) The set of solutions might be three dimensional

A7: The plane $S$ in 3D space is given by $x - 2y + 6z = -4$. Write a parametric form for $S$.

$$(-4, 0, 0) \text{ on plane}$$
$$(2, 1, 0) \text{ and } (0, 3, 1) \text{ are } \perp \text{ to the normal}$$

$$x = (-4, 0, 0) + s(2, 1, 0) + t(0, 3, 1).$$

A8: Consider the system of equations below that has the parameter $a$.

$$
\begin{align*}
&x + y + z = 1 \\
&2x + 4y + 6z = 2a \\
&x + 2y + 3z = 0 \\
&-x + z = 0
\end{align*}
$$

Find a value of $a$ so that the system of equations below has infinitely many solutions or explain why no such value of $a$ exists.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & a \\
1 & 2 & 3 & 0 \\
-1 & 0 & 1 & 0
\end{bmatrix} \sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & a-1 \\
0 & 1 & 2 & 1 \\
0 & 1 & 2 & 1
\end{bmatrix}
$$

system never has solutions.
For questions A9 and A10 below consider the line $L$ and the plane $P$ given below:

$\begin{align*}
L : \quad & \mathbf{x} = [2, 2, -2] + t[1, 2, 3] \\
P : \quad & 4x - 2y = 4
\end{align*}$

A9: Show that $L$ and $P$ are parallel. Include a brief explanation (one sentence) as well as a calculation in your answer.

Plane normal $\mathbf{n} = (4, -2, 0)$  
Line direction $\mathbf{L} = (1, 2, 3)$

$\mathbf{L} \cdot \mathbf{n} = 0 \checkmark$ line and plane normal are perpendicular.

A10: What is the distance between $L$ and $P$?

Note: $(2, 2, -2)$ is on the plane, so $L$ is a subset of $P$, distance is zero.
Test1 C

B1 (Solution)

Consider the linear system below for the unknowns x, y, and z. It is known that the system has a unique solution.

\[
\begin{align*}
  x + 2y + 3z &= 2 \\
  2x + 3y + 4z &= -2 \\
  x + y + 2z &= 41
\end{align*}
\]

(a) (1 mark) write the system in augmented matrix.

\[
\begin{pmatrix}
  1 & 2 & 3 & 2 \\
  2 & 3 & 4 & -2 \\
  1 & 1 & 2 & 41 \\
\end{pmatrix}
\]

(b) (3 mark) Do row operations (Gaussian elimination) on the augmented matrix to change it to echelon form.

\[
\begin{pmatrix}
  1 & 2 & 3 & 2 \\
  2 & 3 & 4 & -2 \\
  1 & 1 & 2 & 41 \\
\end{pmatrix} \equiv \begin{pmatrix}
  1 & 2 & 3 & 2 \\
  0 & -1 & -2 & -6 \\
  0 & -1 & -1 & 39 \\
\end{pmatrix} \equiv \begin{pmatrix}
  1 & 2 & 3 & 2 \\
  0 & 1 & 1 & 45 \\
  0 & 0 & 1 & 45 \\
\end{pmatrix}
\]

(c) (1 mark) Find the solution to the problem from the form above.

From row 3, \( z = 45 \).

From row 2, \( -y - 2z = -6 \) and this gives \( y = -84 \).

From row 1, \( x + 2y + 3z = 2 \), \( x = 35 \).

Therefore, the solution to the system is

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  35 \\
  -84 \\
  45
\end{pmatrix}
\]
B2: A plane $P$ contains the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

(a) [1 mark] Give a parametric equation for $P$.

**Solution:** We are given three points in $P$. A parametric equation requires one point in $P$, and two vectors parallel to $P$. To find the two vectors, we choose two distinct pairs of points, and find the vector between them. For instance, we can take the vector $[1, -2, 0]$ (which has its head at $(1, 0, 0)$ when its tail is at $(0, 1, 0)$) and the vector $[0, 1, -1]$. Then together with a point in the plane, say $(1, 0, 0)$, we find a parametric equation for $P$:

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(b) [2 marks] Give an equation for $P$.

**Solution:** We’ll need a normal vector to $P$, which we can get from the two vectors above that are parallel to $P$.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, our equation has the form $x + y + z = c$ for some constant $c$. To find $c$, we plug in a point from $P$, and see:

$$x + y + z = 1$$

(c) [1 mark] Is the point $(-4, 12, -7)$ in $P$?

**Solution:** Using the equation we found above, with $x = -4$, $y = 12$, and $z = -7$, we see that $x + y + z = -4 + 12 - 7 = 1$, so yes, this point is in the plane.

(d) [1 mark] Is the vector $[-4, 12, -7]$ parallel to $P$?

**Solution:**

- One way to solve this is to note that the point $(-4, 12, -7)$ is in $P$, but the origin is not. So, when the head of the vector $[-4, 12, -7]$ is in $P$, its tail is not. Therefore, the vector is not parallel to the plane.

- Another way to solve this is to note that, if a vector is parallel to $P$, then it is orthogonal to its normal vector. Since $[1, 1, 1] \cdot [-4, 12, -7] \neq 0$, our vector is not parallel to the plane.

- A third way to solve this is to note that a vector parallel to the plane will be a linear combination of the two direction vectors from the parametric equation.
of the plane. The linear system of equations \[
\begin{align*}
1s + 0t &= -4 \\
-1s + 1t &= 12 \\
0s - 1t &= -7
\end{align*}
\] has no solutions. (You can see this by row-reducing the augmented matrix, or by substitution.) So, the vector is not parallel to the plane.

B3: Consider the system of equations below in augmented matrix form:
\[
\begin{bmatrix}
3 & 2 & 3 & | & 2 \\
2 & 1 & 1 & | & 3
\end{bmatrix}
\]

(a) [1 mark] Give a brief geometrical description of the problem described by the system.
**Solution:** The intersection of two planes in \( \mathbb{R}^3 \).

(b) [1 mark] Put the augmented matrix into reduced row echelon form.
**Solution:** There are many ways to do the row-reduction, but the end matrix should be the same as shown.

\[
\begin{align*}
R_1 - R_2 &: \\
&\begin{bmatrix}
1 & 1 & 2 & | & -1 \\
2 & 1 & 1 & | & 3
\end{bmatrix}
\end{align*}
\]

\[
R_2 \rightarrow R_2 - 2R_1 \\
\begin{bmatrix}
1 & 1 & 2 & | & -1 \\
0 & -1 & -3 & | & 5
\end{bmatrix}
\]

\[
R_1 \rightarrow R_1 + R_2 \\
\begin{bmatrix}
1 & 0 & -1 & | & 4 \\
0 & 1 & 3 & | & 5
\end{bmatrix}
\]

\[
R_2 \rightarrow -R_2 \\
\begin{bmatrix}
1 & 0 & -1 & | & 4 \\
0 & 1 & 3 & | & -5
\end{bmatrix}
\]

(c) [1 mark] Find the solutions or solutions, if any.
**Solution:** If we let the third variable be the parameter \( s \), then our first equation tells us \( x_1 - s = 4 \), hence \( x_1 = 4 + s \). The second equation tells us \( x_2 + 3s = -5 \), hence \( x_2 = -5 - 3s \). So, our solutions are:

\[
x = \begin{bmatrix}
4 \\
-5 \\
0
\end{bmatrix} + s \begin{bmatrix}
1 \\
-3 \\
1
\end{bmatrix}
\]

(d) [1 mark] Give a geometrical description of the solution set.
**Solution:** It is a line in \( \mathbb{R}^3 \), passing through the point \((4, -5, 0)\), in the direction of \([1, -3, 1]\).

(e) [1 mark] Repeat (a)-(d) for the system below:
\[
\begin{bmatrix}
3 & 2 & | & 2 \\
2 & 1 & | & 3 \\
3 & 1 & | & 1
\end{bmatrix}
\]
**Solution:** Geometrically, this is the intersection of three lines in $\mathbb{R}^2$.

\[
\begin{bmatrix}
3 & 2 & 2 \\
2 & 1 & 3 \\
3 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
2 & 1 & 3 \\
3 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
2 & 1 & 3 \\
1 & 0 & -2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & -2 \\
2 & 1 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & -2 \\
2 & 1 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & -2 \\
2 & 0 & 2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 6 \\
\end{bmatrix}
\]

There is no solution to this system of equations. There is no point that is on all three lines.