Part A - Short Answer Questions, 1 mark each

For Questions A1-A2 below, let

\[ \mathbf{a} = [1, 1, 5] \]

A1: Find the length of \( \mathbf{a} \).

\[ \sqrt{27} \]

A2: Find a unit vector in the direction \( \mathbf{a} \). Your answer should be in the form \([x, y, z] \) with \( x \), \( y \), and \( z \) determined.

\[ \left[ \frac{1}{\sqrt{27}}, \frac{1}{\sqrt{27}}, \frac{5}{\sqrt{27}} \right] \]

A3: Compute the determinant of the matrix below:

\[
\begin{vmatrix}
1 & 2 & 0 \\
1 & 0 & 1 \\
7 & 6 & 5 \\
\end{vmatrix}
= 1(-6) - 2(5-7) = -6 + 4 = -2
\]

A4: Circle all statements below that are true for all nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( \mathbb{R}^3 \):

(a) \( \mathbf{a} \cdot \mathbf{a} \neq 0 \)
(b) \( \mathbf{a} \cdot \mathbf{b} \neq 0 \)
(c) \( \mathbf{a} \times \mathbf{b} \neq 0 \)
(d) \( \{\mathbf{a}, \mathbf{b}\} \) is a linearly independent set
(e) \( \mathbf{b} \times \mathbf{b} = 0 \)

A5: Express \( \mathbf{v} = [3, -1] \) as a linear combination of vectors \( \mathbf{a} = [1, 1] \) and \( \mathbf{b} = [2, -3] \).

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & -3 & 1 \\
\end{bmatrix}
\mathbf{t}
+ \begin{bmatrix}
1 & 2 & 3 \\
0 & -5 & -4 \\
\end{bmatrix}
\mathbf{s}
= \begin{bmatrix}
\frac{4}{5} \\
\frac{1}{5} \\
\end{bmatrix}
\]

\[
\mathbf{v} = \frac{4}{5} \mathbf{a} + \frac{1}{5} \mathbf{b}^2
\]
A6: Consider the following lines of MATLAB code:

\[
A = \text{zeros}(2,4);
A(1,2) = 5;
A(2,1:3) = [1 2 3];
\]

\[
\begin{bmatrix}
0 & 5 & 0 & 0 \\
1 & 2 & 3 & 0
\end{bmatrix}
\]

What is the result in \( A \)?

A7: Let \((0,5), (1,-4), (2,8)\) be 3 points on the graph of the function \( y = f(x) = ax^2 + bx + c \). Find the function \( f(x) \).

\[
\begin{aligned}
&c = 5 \\
&5 + a + b = -4 \\
&5 + 4a + 2b = 8
\end{aligned}
\]

\[
\begin{bmatrix}
1 & 1 & -9 \\
4 & 2 & -3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -9 \\
0 & -2 & 3a
\end{bmatrix}
\]

\[
b = \frac{-3a}{2}, \quad a = \frac{21}{2}
\]

A8: Let \( a = [1,1,1], b = [-3,2,-1] \). Find \( b_\perp \) and \( b_\parallel \) such that \( b = b_\perp + b_\parallel \), where \( b_\perp \) is perpendicular to \( a \) and \( b_\parallel \) is parallel to \( a \).

\[
b_\parallel = \text{proj}_a b = \frac{-2}{3} (1,1,1) = \left( \frac{-2}{3}, \frac{-2}{3}, \frac{-2}{3} \right)
\]

\[
b_\perp = b - b_\parallel = (-3,2,-1) - \left( \frac{-2}{3}, \frac{-2}{3}, \frac{-2}{3} \right)
\]

\[
= \left( \frac{-7}{3}, \frac{8}{3}, \frac{-1}{3} \right)
\]
Questions A9-A10 below concern the point \( Q \) and plane \( P \) described below:

\[
Q : (3, 4, 2) \\
P : 2x - y + z - 6 = 0
\]

**A9:** Write a parametric form for the line passing through \( Q \) that is perpendicular to \( P \).

\[
\mathbf{l} : (3, 4, 2) + t(2, -1, 1) = (3 + 2t, 4 - t, 2 + t).
\]

**A10:** What is the distance from \( Q \) to \( P \)?

Let \( R \) be the point on \( \mathbf{l} \) that intersects \( P \).

\[
2(3 + 2t) - (4 - t) + (1 + 2t) - 6 = 0 \\
9t - 3 = 0 \\
t = \frac{1}{3}.
\]

\[
R : \left( \frac{4}{3}, \frac{11}{3}, \frac{5}{3} \right).
\]

The distance \( d \) is the length of \( QR \):

\[
d = \sqrt{\left( \frac{4}{3} - 3 \right)^2 + \left( \frac{11}{3} - 4 \right)^2 + \left( \frac{5}{3} - 2 \right)^2} \\
= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{3}}.
\]
Consider the linear system below for the unknowns $x, y,$ and $z$. It is known that the system has a unique solution.

$$\begin{align*}
  x + y + z &= 2 \\
  2x + 3y + 4z &= 3 \\
  x + 2y + z &= 4
\end{align*}$$

(a) (1 mark) Write the system in augmented matrix.

$$\begin{pmatrix}
  1 & 1 & 1 & : & 2 \\
  2 & 3 & 4 & : & 3 \\
  1 & 2 & 1 & : & 4
\end{pmatrix}$$

(b) (3 marks) Do row operations (Gaussian elimination) on the augmented matrix to change it to echelon form.

(augmented matrix $\equiv$)

$$\begin{pmatrix}
  1 & 1 & 1 & : & 2 \\
  0 & 1 & 2 & : & -1 \\
  0 & 0 & -2 & : & 3
\end{pmatrix}$$

(c) (1 mark) Find the solution to the problem from the form above.

From row 3, $z = -\frac{3}{2}$.

From row 2, $y + 2z = -1$ and this gives $y = 2$.

From row 1, $x + y + z = 2$, $x = -\frac{3}{2}$.

Therefore, the solution to the system is

$$\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  \frac{3}{2} \\
  2 \\
  -\frac{3}{2}
\end{pmatrix}$$
B2: Consider the linear system below with parameters c and d
\[
\begin{align*}
  x - 2y + 3z &= 4, \\
  x + cy - z &= d, \\
  2x - 2z &= 0.
\end{align*}
\]

(a) [2 points] For what value or values of c and d does the system have a unique solution?
(b) [1] For what value or values of c and d does the system have an infinite number of solutions?
(c) [2] Take c = 1 and d = 2. Write the reduced row echelon form of the system in an augmented matrix.

Solution:

(a)
\[
[A|\vec{b}] = \begin{bmatrix}
1 & -2 & 3 & 4 \\
1 & c & -1 & d \\
2 & 0 & -2 & 0
\end{bmatrix}
\]

(1) = (2) - (1)
(2) = (3) - 2(1)
(3) = (3) - 2(1)

= \begin{bmatrix}
1 & -2 & 3 & 4 \\
0 & c + 2 & -4 & d - 4 \\
0 & 4 & -8 & -8
\end{bmatrix}

Thus, if \(c \neq 0\) there is a unique solution.

(b) If \(c = d = 0\), then there exist infinitely many solutions.

(c)
\[
\begin{bmatrix}
1 & -2 & 3 & 4 \\
0 & 1 & -2 & -2 \\
0 & 0 & 2 & 4
\end{bmatrix}
\]

(1) = (1) + 2(2)
(2) = (3) - 2(2)
(3) = (3) - 2(2)

= \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -2 & -2 \\
0 & 0 & 1 & 2
\end{bmatrix}

(1) = (1) + (3)
(2) = (2) + 2(3)
(3) = (3) + 2(3)

Thus, if \(c \neq 0\) there is a unique solution.
B3: Let line $L$ be described in parametric form by

$$L : \vec{x} = [0, -1, 1] + t[1, 1, -1],$$

and two parallel planes $S_0$ and $S_1$ be described by equations

$$\begin{cases} S_0 : x + 2y - z = 0, \\ S_1 : x + 2y - z = 6, \end{cases}$$

(a) [2 marks] Find the point of intersection, $P_0$, between $L$ and $S_0$.

(b) [1] Find the point of intersection, $P_1$, between $L$ and $S_1$.

(c) [2] Find a second point, $P_2$, on $S_1$ such that $P_0, P_1, P_2$ form a right triangle.

Solution:

(a) $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t-1 \\ 1-t \end{bmatrix}$. Substitute into the equation for $S_0$:

$$t + 2(t-1) - (1-t) = 0 \Rightarrow 4t = 3 \Rightarrow t = \frac{3}{4}.$$ Thus,

$$P_0 : \quad \vec{p}_0 = \vec{x}(t = \frac{3}{4}) = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

(b) Now, substitute the line equation into equation for $S_1$:

$$t + 2(t-1) - (1-t) = 6 \Rightarrow 4t = 9 \Rightarrow t = \frac{9}{4}.$$ Thus,

$$P_1 : \quad \vec{p}_1 = \vec{x}(t = \frac{9}{4}) = \begin{bmatrix} \frac{9}{4} \\ -\frac{9}{4} \\ \frac{5}{4} \end{bmatrix}.$$

(c) Draw another line, $L_1$, starting from $P_0$ on $S_0$ and following the normal direction of the two planes $\vec{n} = [1, 2, -1]$. Thus, the equation for $L_1$ is
\[
\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} s + \frac{3}{4} \\ 2s - \frac{1}{4} \\ -s + \frac{1}{4} \end{bmatrix}.
\]

The point of intersection between \(L_1\) and \(S_1\) is \(P_2\). Substitute \(L_1\) equation into the equation for \(S_1\):

\[
s + \frac{3}{4} + 2(2s - \frac{1}{4}) - (-s + \frac{1}{4}) = 6 \quad \Rightarrow \quad 6s = 6 \quad \Rightarrow \quad s = 1.
\]

Thus,

\[
P_2 : \quad \vec{p}_2 = \vec{x}(s = 1) = \begin{bmatrix} 1 + \frac{3}{4} \\ 2 - \frac{1}{4} \\ -1 + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{7}{4} \\ \frac{3}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 \\ 7 \\ -3 \end{bmatrix}.
\]